ASYMPTOTIC STUDY OF THE WAVE FIELD GENERATED BY A STEADILY MOVING PULSATING SOURCE

by

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In the past, the problem of movements of ships in operation has interested numerous writers. Thus M.D. HASKIND (1946), R. BRARD (1948) and T. HANAOKA (1953), then L.N. STRETENSKII (1954) and T.H. HAVELOCK (1958) have given different formulations of the GREEN function relative to this problem when the depth is infinite. In the more delicate case of a finite water depth, suggested formulations are less common ; the first belonging to J.K. LUNDE (1951) was improved by J.C. DERN and G. FERNANDEZ (1979).

The analysis of the asymptotic wave field was outlined from these formulations by M.J. LIGHTHILL (1956), K. EGGERS (1957), and E. BECKER (1958).

In this article we present the asymptotic expansion of the GREEN function and the wave far field velocity for each case linked to the values of the pulsation and velocity, together with the radiation condition at infinity in the case of unlimited depth. Afterwards, we envisage the case of finite depth and illustrate the qualitative conservation of the results acquiered previously. For this purpose we use the GREEN function formulations proposed by P. GUEVEL, J. BOUGIS and D.C. HONG (1979) and which are adapted to the demands of numerical treatment and asymptotic expansions.

Let us take a solid with a horizontal constant mean speed in an incompressible perfect fluid subject to the action of gravity, limited by a free surface and by a level and horizontal bottom ; it oscillates around its mean position when it is affected by an incident swell.

The introduction of the classic linearity hypothesis has two basic consequences. The first one is the possibility of superposing different states obtained separately. The second is to write boundary conditions on the mean positions of the free surface and the hull.

In these conditions, the flow around the body can be determined using elementary operators (GREEN function) which generate in the fluid a harmonic potential satisfying, by construction, the linearised free surface condition, the condition of slipping on the bottom and the condition of radiation to infinity.

The last constraint is expressed simply in the case of the wave resistance in still water by writing that the fluid is not disturbed infinitely upstream, and in the case of diffraction radiation at zero FROUDE number by suppressing the regressive gravity waves generated at infinity.

In the more complex case of diffraction radiation with forward speed, the condition radiation at infinity is not so evident due to the diversity of gravity wave systems which appear. We therefore prefer not to introduce solutions foreign to the penomenon studied, preserving qualitatively the fluid's dissipative character by a time constant ε which we shall tend to zero after determining the solution. This classic procedure constitutes a mathematical artifice which takes into account the progressive nature of the setting in motion of the physical fluid and the tendency of the latter to develop towards a state of static equilibrium when all disturbances have ceased.

The asymptotic analysis of velocity potential generated by a source type elementary operator allows us to establish the topography of the wave far field and to determine the radiation condition which will traduce the conformity of the solution obtained without e to physical reality by elimination of aberrant solutions.

I BOUNDARY PROBLEM

The determination of the velocity potential generated by a pontual source type operator, moving with a constant uniforme speed $U.\vec{i}_x$ and whose strenght is the sinusoidal time function Q.cos ωt, is reduced, given the hypotheses agreed upon, to the solution of a boundary problem ; which can be expressed in a fixed frame, or in a system of axis moving with the source (figure 1).

Figure 1

1,1 Absolute potential expressed in the fixed frame

In the fixed frame (0 ; x, y, z) the absolute potential is the solution of the following boundary problem :

$$\left.\begin{array}{ll} \Delta\phi(M;t) = 0 & \forall \ M \in \mathfrak{D} - M' \\ \frac{\partial^2}{\partial t^2}\phi(M;t) + 2\varepsilon \frac{\partial}{\partial t}\phi(M;t) + g\frac{\partial}{\partial z}\phi(M;t) = 0 & \forall \ M \in SL \\ \frac{\partial}{\partial z}\phi(M;t) = 0 & \forall \ M \in F \\ \lim\limits_{|MM'| \to \infty} \phi(M;t) = 0 & \forall \ M \in \mathfrak{D} \end{array}\right] \quad (1,1)$$

1,2 Absolute potential expressed in the mobile frame

In the mobile frame $(O_1\ ;\ X_1,\ y_1,\ z_1)$ the absolute potential is the solution of the following boundary problem :

$$\left.\begin{array}{ll}
\Delta\phi(M;t) = 0 & \forall\ M \in \mathfrak{D} - M' \\
\dfrac{\partial^2}{\partial t_1^2}\phi(M;t) - 2U\dfrac{\partial^2}{\partial t_1 \partial x_1}\phi(M;t) + U^2\dfrac{\partial^2}{\partial x_1^2}\phi(M;t) & \\
+ 2\varepsilon\dfrac{\partial}{\partial t_1}\phi(M;t) - 2U\varepsilon\dfrac{\partial}{\partial x_1}\phi(M;t) + g\dfrac{\partial}{\partial z_1}\phi(M;t) = 0 & \forall\ M \in SL \\
\dfrac{\partial}{\partial z_1}\phi(M;t) = 0 & \forall\ M \in F \\
\lim\limits_{|MM'|\to\infty} \phi(M;t) = 0 & \forall\ M \in \mathfrak{D}
\end{array}\right] \qquad (1,2)$$

1,3 Construction principle of the speed potential function

The free surface condition brings in partial derivatives in relation to the two independent variables z and t. We therefore have recourse to an integral formulation to transform the derivation relative to z into an algebraic operator and thus obtain a linear differential equation of the second order the only derivations of which are temporal in the absolute reference.

We shall therefore construct the velocity potential function in the form (1,3) in the case of infinite depth, and in the form (1,4) in the case of finite depth.

$$\phi\ (M;t) = -\frac{Q}{4\pi}\left[\frac{\cos \omega t}{|MM'|} + \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\ \mathrm{Re}\left\{\int_{-\pi/2}^{+\pi/2} d\theta \int_0^\infty \hat{g}(\theta,k,\varepsilon;t)e^{kz}\right.\right.$$

$$\left.\left. e^{ik\ (x\cos\theta + y\sin\theta)}\ k\ dk\right\}\right] \qquad (1,3)$$

$$\phi(M;t) = -\frac{Q}{4\pi}\left[\frac{\cos\omega t}{|MM'|} + \frac{\cos\omega t}{|MM'_1|} + \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \mathrm{Re}\left\{\int_{-\pi/2}^{+\pi/2} d\theta \int_0^\infty \hat{g}(o,k,\varepsilon;t) \right.\right. \tag{1,4}$$

$$\left.\left. \frac{\operatorname{ch} k(z+h)}{\operatorname{ch} kh}\, e^{ik(x\cos\theta + y\sin\theta)}\, k\, dk\right\}\right]$$

expression in which M'_1 denotes the symmetrical point of M' in relation to the bottom.

At this stage we draw attention to the following three remarks :

- These functions are harmonic in $\mathfrak{D}$ - M' by construction since $\frac{1}{|MM'|}$ and $\frac{1}{|MM'_1|}$ are harmonic, and since the functions $e^{kz}\, e^{ik(x\cos\theta + y\sin\theta)}$ and $\operatorname{ch} k(z+h)\, e^{ik(x\cos\theta + y\sin\theta)}$ are elementary solutions to the Laplace equation in the fluid areas defined respectively by $z < 0$ and $-h < z < 0$, obtained by variables separation procedure.

- These functions verify by construction $\frac{\partial}{\partial z}\phi(M;t) = 0$ on the bottom or $\lim_{z\to -\infty} \phi(M;t) = 0$ when the depth is unlimited (a condition which takes the place of the preceding one in this case) ;

- These functions must verify the condition $\lim_{|MM'|\to\infty} \phi(M;t) = 0\ \forall M \in \mathfrak{D}$ if we do not want to write non-convergent integrals. Nevertheless we shall have to check later that the solutions found conform to this demand.

In these conditions, the functions $\hat{g}(\theta,k,\varepsilon;t)$ will be determined by imposing on $\phi(M;t)$ to satisfy the linearised free surface equation.

II THE DEPTH IS UNLIMITED

When the depth is infinite, the velocity potential function, obtained by the procedure previously explained, is written in integral form :

$$\phi(M;t) = \phi_0(M;t) + \phi_1(M;t) + \phi_2(M;t) \qquad (2,1)$$

with :

$$\phi_0(M;t) = -\frac{Q}{4\pi}\left[\frac{1}{|MM'|} - \frac{1}{|MM'_1|}\right]\cos\omega t \qquad (2,2)$$

$$\phi_1(M;t) = -\frac{Q}{4\pi}\left[\lim_{\tilde{\varepsilon}\to 0+} -\frac{1}{\pi\ell}\,\mathrm{Re}\left\{e^{i\omega t}\int_{-\pi/2}^{+\pi/2} d\theta\int_0^{\infty}\frac{e^{K(Z+Z'+i\Omega)}\,K\,dK}{(\tilde{\omega}-FK\cos\theta)^2-K-2i\tilde{\varepsilon}(\tilde{\omega}-FK\cos\theta)}\right\}\right] \qquad (2,3)$$

$$\phi_2(M;t) = -\frac{Q}{4\pi}\left[\lim_{\tilde{\varepsilon}\to 0+} -\frac{1}{\pi\ell}\,\mathrm{Re}\left\{e^{-i\omega t}\int_{-\pi/2}^{+\pi/2} d\theta\int_0^{\infty}\frac{e^{K(Z+Z'+i\Omega)}\,K\,dK}{(\tilde{\omega}+FK\cos\theta)^2-K+2i\tilde{\varepsilon}(\tilde{\omega}+FK\cos\theta)}\right\}\right] \qquad (2,4)$$

Expressions in which M'_1 denotes the symmetrical point of M' in relation to the free surface, and in which the characteristic sizes of flow are adimensionalised in relation to a reference length ℓ which is, for example, the immersion depth of the source. Thus we have :

$$X = \frac{x}{\ell} \qquad Y = \frac{y}{\ell} \qquad Z = \frac{z}{\ell} \qquad \nu = \tilde{\omega} F$$

$$K = k\ell \qquad \tilde{\omega} = \omega\sqrt{\frac{\ell}{g}} \qquad \tilde{\varepsilon} = \varepsilon\sqrt{\frac{\ell}{g}} \qquad F = \frac{U}{\sqrt{g\ell}} \qquad (2,5)$$

$$\Omega = (X - X')\cos\theta + (Y - Y')\sin\theta$$

Let K_1 and K_2 $(0 < K_1 < K_2)$ be the poles of the integrand of $\phi_1(M;t)$ and K_3 and K_4 ($0 < K_3 < K_4$ if K_3 and K_4 are real ; $\mathrm{Im}(K_3) < 0 < \mathrm{Im}(K_4)$ if K_3 and K_4 are complex) those of the integrand of $\phi_2(M;t)$.

The preceding functions are written after integration relative to K :

$$\phi_1 (M;t) = -\frac{Q}{4\pi^2 \ell} \, \mathrm{Re} \left\{ e^{i\omega t} \int_{-\pi/2}^{+\pi/2} \frac{K_1 \, G1 (K_1\zeta) - K_2 \, G1 (K_2\zeta)}{\sqrt{1 + 4 \nu \cos \theta}} \, d\theta \right\} \quad (2,6)$$

$$\phi_2 (M;t) = -\frac{Q}{4\pi^2 \ell} \, \mathrm{Re} \left\{ e^{-i\omega t} \left[\int_{-\pi/2}^{-\theta_c} \frac{K_3 \, G3 (K_3\zeta) - K_4 \, G1 (K_4\zeta)}{\sqrt{1 - 4 \nu \cos \theta}} \, d\theta \right. \right.$$

$$- \int_{-\theta c}^{-\theta'_c} i \, \frac{K_3 \, G1 (K_3\zeta) - K_4 \, G3 (K_4\zeta)}{\sqrt{4 \nu \cos \theta - 1}} \, d\theta$$

$$- \int_{-\theta'_c}^{+\theta'_c} i \, \frac{K_3 \, G2 (K_3\zeta) - K_4 \, G2 (K_4\zeta)}{\sqrt{4 \nu \cos \theta - 1}} \, d\theta$$

$$- \int_{\theta'_c}^{\theta c} i \, \frac{K_3 \, G1 (K_3\zeta) - K_4 \, G3 (K_4\zeta)}{\sqrt{4 \nu \cos \theta - 1}} \, d\theta$$

$$\left. \left. + \int_{\theta c}^{\pi/2} \frac{K_3 \, G3 (K_3\zeta) - K_4 \, G1 (K_4\zeta)}{\sqrt{1 - 4 \nu \cos \theta}} \, d\theta \right] \right\} \quad (2.7)$$

Expressions in which $\cos \theta_c$ is equal to $\frac{1}{4\nu}$ and $\cos \theta'_c$ is equal to $\frac{1}{2\nu}$. Thevariable ζ denotes $z + z' + i \Omega$, and the functions Gl, G2 and G3 represent the following modified exponential intégrale functions :

$$G1 (\xi) = e^{\xi} \, \mathcal{E}_1 (\xi) \qquad 0 < \mathrm{Arg} (\xi) < 2\pi$$

$$G2 (\xi) = e^{\xi} \, E_1 (\xi) \qquad -\pi < \mathrm{Arg} (\xi) < +\pi$$

$$G3 (\xi) = e^{\xi} \, [\mathcal{E}_1 (\xi) + 2i\pi] \qquad 0 < \mathrm{Arg} (\xi) < 2\pi \qquad (2.8)$$

with :

$\mathcal{E}_1(\xi) = E_1(\xi)$ Im $(\xi) > 0$

$\mathcal{E}_1(\xi) = E_1(\xi + i\varepsilon)$ Im $(\xi) = 0$

$\mathcal{E}_1(\xi) = E_1(\xi) - 2i\pi$ Im $(\xi) < 0$ (2,9)

2,1 Near potential and far potential

The asymptotic behaviour of the integral exponential function $[e^{\xi} E_1(\xi) = \xi^{-1} + 0(\xi^{-1})$ when $|\xi| \to \infty]$ enables us to separate terms which decrease a priori, such as ξ^{-1}, from the others. The first are generally called near potential and the second ones far potential. It should be noted that the so-called far potential may still contain decreasing terms such as ξ^{-1}. We shall be interested only in far potential here.

Let 3 be the angle for which Ω is cancelled. We thus have : $\Omega = R \sin(\theta - \beta)$ supposing $R = \sqrt{(x - x)^2 + (y - y')^2}$ and $\beta = \alpha - \frac{\pi}{2}$ with $tg\alpha = \frac{y - y'}{x - x'}$.
If we notice that the complex poles (K_3 and K_4 for $-\theta_c < \theta < \theta_c$) lead to real exponential terms the exponent of which is, to the nearest real positive function, - R their study will be superfluous since they will not contribute to the far potential. We then have :

$$\phi_{11}(M;t) = \frac{Q}{2\pi\ell} \operatorname{Re}\left\{ ie^{i\omega t}\left[\int_{-\pi/2}^{\beta} \frac{K_1 e^{K_1\zeta} d\theta}{\sqrt{1+4\nu\cos\theta}} - \int_{-\pi/2}^{\beta} \frac{K_2 e^{K_2\zeta} d\theta}{\sqrt{1+4\nu\cos\theta}}\right]\right\} \quad \forall \nu \quad (2,10)$$

$$\phi_{21}(M;t) = -\frac{Q}{2\pi\ell} \operatorname{Re}\left\{ ie^{-i\omega t}\left[\int_{\beta}^{+\pi/2} \frac{K_3 e^{K_3\zeta} d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{-\pi/2}^{\beta} \frac{K_4 e^{K_4\zeta} d\theta}{\sqrt{1-4\nu\cos\theta}}\right]\right\} \quad \nu < \frac{1}{4} \quad (2,11)$$

$$\phi_{21}(M;t) = -\frac{Q}{2\pi\ell} \operatorname{Re}\left\{ ie^{-i\omega t}\left[\int_{\beta}^{-\theta_c} \frac{K_3 e^{K_3\zeta} d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{\theta_c}^{\pi/2} \frac{K_3 e^{K_3\zeta} d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{-\pi/2}^{\beta} \frac{K_4 e^{K_4\zeta} d\theta}{\sqrt{1-4\nu\cos\theta}}\right]\right\} \quad \nu > \frac{1}{4} \; ; \; -\frac{\pi}{2} < \beta < -\theta_c \quad (2,12)$$

$$\phi_{21}(M;t) = -\frac{Q}{2\pi\ell}\,\mathrm{Re}\left\{ ie^{-i\omega t}\left[\int_{\theta_c}^{\pi/2}\frac{K_3\, e^{K_3\zeta}\, d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{-\pi/2}^{-\theta_c}\frac{K_4\, e^{K_4\zeta}\, d\theta}{\sqrt{1-4\nu\cos\theta}}\right]\right\} \quad \nu>\nu_c\ ;|\beta|<|\theta_c| \quad (2,13)$$

$$\phi_{21}(M;t) = -\frac{Q}{2\pi\ell}\,\mathrm{Re}\left\{ ie^{-i\omega t}\left[\int_{\beta}^{\pi/2}\frac{K_3\, e^{K_3\zeta}\, d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{-\pi/2}^{-\theta_c}\frac{K_4\, e^{K_4\zeta}\, d\theta}{\sqrt{1-4\nu\cos\theta}} + \int_{\theta_c}^{\beta}\frac{K_4\, e^{K_4\zeta}\, d\theta}{\sqrt{1-4\nu\cos\theta}}\right]\right\} \quad \nu>\nu_c\ ;\ \theta_c<\beta<\frac{\pi}{2} \quad (2,14)$$

All the integrals so defined can be expressed in the following form :

$$I(R) = \int_a^b f(\theta,R)\, e^{iRg(\theta)}\, d\theta \qquad (2.15)$$

Expression in which R represents the projection of $|MM'|$ on a horizontal plane and increases indefinitely when the asymptotic analysis is carried out ; it follows that $e^{iRg(\theta)}$ oscillates very rapidly in the interval (a,b). The function f (θ,R) is continuous, limited and does not oscillate rapidly in the interval considered. We are therefore in the conditions to apply the stationary phase principle (Cauchy - Lord Kelvin). According to the values of the parameters β and ν the functions $g_i(\theta)$ corresponding respectively to the poles K_i $(i\in[1,4])$ possess or do not possess minima, maxima or horizontal tangent inflection points, which determine the behaviour of $\phi(M;t)$ and of its derivatives to infinity. We must therefore study the existence of zeros of the functions $g'_i(\theta)$ and the corresponding signs for $g''_i(\theta)$.

2,2 Asymptotic expansion of $\phi_{11}(M;t)$

The function $g_1(\theta)$ relative to the first pole is expressed :

$$g_1(\theta) = K_1(\theta)\,\sin(\theta-\beta) = \frac{1+2\nu\cos\theta - \sqrt{1+4\nu\cos\theta}}{2F^2\cos^2\theta}\,\sin(\theta-\beta) \qquad (2,16)$$

and admits as derivative with respect to :

$$g'_1(\theta) = \frac{2\nu^2\cos\beta}{F^2\left[1+4\nu\cos\theta+(1+2\nu\cos\theta)\sqrt{1+4\nu\cos\theta}\right]\cos\theta}\left[\frac{1}{2}\left(\frac{\cos(2\theta-\beta)}{\cos\beta} - 1\right)+\sqrt{1+4\nu\cos\theta}\right] \quad (2,17)$$

The first factor of $g'_1(\theta)$ is never cancelled. Our study is thus led to that of zeros of the second factor except for the two integration area limits for which a local study proves necessary.

When θ tends to $-\frac{\pi}{2}$, $g'_1(\theta)$ has a limited expansion in relation to $(\theta + \frac{\pi}{2})$. The latter is written in the vicinity of $-\frac{\pi}{2}$:

$$g'_1(\theta) = \varpi^2\left[2\nu\cos\beta-\sin\beta\right] + \varpi^2\left[(1-10\nu^2)\cos\beta+4\nu\sin\beta\right](\theta+\frac{\pi}{2}) + o(\theta+\frac{\pi}{2}) \quad (2,18)$$

When θ tends to β, $g'_1(\theta)$ admits as limit $K_1(\beta)$, a value included in the interval $]0,\varpi^2]$ and therefore strictly positive.

In the open interval $]-\frac{\pi}{2},\beta[$, a graphic study will enable us to conclude on the existences of zeros as well as on the increase or decrease of $g'_1(\theta)$ in the vicinity of these values (figure 2).

Figure 2

The results obtained have been brought together on figure 8. The angle β_1 is defined by the relation $tg\beta_1 = 2\nu$, and the angle β'_1 by the existence of θ'_1 such as $g'_1(\theta'_1) = g''_1(\theta'_1) = 0$. Note that for $\nu = \frac{1}{\sqrt{2}}$, the angles β_1 and β'_1 are equal.

The function $g_2(\theta)$ relative to the second pole is expressed :

$$g_2(\theta) = K_2(\theta)\ \sin(\theta-\beta) = \frac{1+2\nu\cos\theta + \sqrt{1+4\nu\cos\theta}}{2F^2\cos^2\theta}\ \sin(\theta-\beta) \qquad (2.19$$

and admits as derivative with respect to :

$$g_2'(\theta) = \frac{2\nu^2\cos\beta}{F^2\left[1+4\nu\cos\theta-(1+2\nu\cos\theta)\sqrt{1+4\nu\cos\theta}\right]\cos\theta}\left[\frac{1}{2}\left(\frac{\cos(2\theta-\beta)}{\cos\beta}-1\right)-\sqrt{1+4\nu\cos\theta}\right] \qquad (2,20$$

As previously, the study of the zeros of $g'_2(\theta)$ is reduced to that of the term in square brackets except for the two area limits.

When θ tends to $-\frac{\pi}{2}$, $g'_2(\theta)$ has a limited expansion in relation to $(\theta+\frac{\pi}{2})$. The latter is written in the vicinity of $-\frac{\pi}{2}$:

$$g_2'(\theta) = \frac{1}{F^2(\theta+\frac{\pi}{2})^3}\left[2\cos\beta+(\sin\beta+2\nu\cos\beta)(\theta+\frac{\pi}{2}) + o(\theta+\frac{\pi}{2})\right] \qquad (2,21$$

When θ tends to β, $g'_2(\theta)$ is identified with $K_2(\beta)$ which is always strictly positive.

In the open interval $]-\frac{\pi}{2}\ \beta[$, a graphic study enables us to conclude again :

Figure 3

The results obtained are shown together on figure 8. The angle β_2 defined by the existence of θ_2 such as $g'_2(\theta_2) = g''_2(\theta_2) = 0$ is always greater than Kelvin's angle β_K ($tg\beta_K = \sqrt{8}$), and tends to $\pi/2$ when ν tends to infinity.

2,3 Asymptotic expansion of $\phi_{21}(M;t)$:

The study of $\phi_{21}(M;t)$ is more complex than the foregoing study since there obviously exist two distinct modes according to whether ν is smaller or greater than $\nu_c = \frac{1}{4}$.

The function $g_3(\theta)$ relative to the third pole is expressed :

$$g_3(\theta) = K_3(\theta) \sin(\theta-\beta) = \frac{1-2\nu\cos\theta + \sqrt{1-4\nu\cos\theta}}{2F^2\cos^2\theta} \sin(\theta-\beta) \quad (2,22)$$

for $(1 - 4\nu \cos\theta > 0)$ and admits as derivative :

$$g_3'(\theta) = \frac{2\nu^2\cos\beta}{F^2\left[1-4\nu\cos\theta+(1-2\nu\cos\theta)\sqrt{1-4\nu\cos\theta}\right]\cos\theta}\left[\frac{1}{2}\left(\frac{\cos(2\theta-\beta)}{\cos\beta} - 1\right)+\sqrt{1-4\nu\cos\theta}\right] \quad (2,23)$$

We are led to study the zeros of the second factor except for the limits $\beta, \pi/2$ and possibly $\pm\theta_c$ for which we shall have to carry out a local study.

When θ tends to $\pi/2$, $g'_3(\theta)$ has a limited expansion whith respect to $(\pi/2 - \theta)$. The latter is written in the vicinity of $\pi/2$:

$$g_3'(\theta) = -\tilde{\omega}^2\left[2\nu\cos\beta-\sin\beta\right] + \tilde{\omega}^2\left[(1-10\nu^2)\cos\beta+4\nu\sin\beta\right]\left(\frac{\pi}{2} -\theta\right) + o\left(\frac{\pi}{2} -\theta\right) \quad (2,24)$$

When θ tends to β, $g'_3(\theta)$ admits as limit $K_3(\beta)$ which is always included in the interval $[\tilde{\omega}^2, 4\tilde{\omega}^2]$ and therefore always positive.

If ν is smaller than ν_c, we shall carry out again a graphic study, but in the interval $]\beta,\pi/2[$. Figure 4 enables us to discuss the existence of zeros of $g'_3(\theta)$ in relation to ν and β.

Figure 4

If ν is greater than ν_c, $g'_3(\theta)$ has a limited expansion in the vicinities of $-\theta_c$ and $+\theta_c$.

$$g'_3(\theta) = 4\tilde{\omega}^2\left[1 - \frac{tg\theta_c}{2\sqrt{\nu\sin\theta_c}}\right]\sqrt{-\theta_c-\theta} + o(\sqrt{-\theta_c-\theta}) \qquad (2,25)$$

$$g'_3(\theta) = 4\tilde{\omega}^2\left[1 + \frac{tg\theta_c}{2\sqrt{\nu\sin\theta_c}}\right]\sqrt{\theta-\theta_c} + o(\sqrt{\theta-\theta_c}) \qquad (2,26)$$

where θ_c is defined by $1 - 4\nu\cos\theta_c = 0$. In these two cases $g'_3(\theta)$ clearly possesses a non zero limit independent of ν and β.

A graphic study will be carried out in the intervals in which the two functions are defined (figure 5).

Figure 5.

The results obtained are shown together in figure 8. The angle β is defined by $tg\beta_3 = 2\nu$ and is therefore identical to β_1, and the angle β'_3 by the existence of θ'_3 such as $g'_3(\theta'_3) = g''_3(\theta'_3) = 0$; $\beta'_3 = 0$ for $\nu = \sqrt{2/27} = 0.272$.

It should be noted that for $\nu = \frac{1}{2\sqrt{3}}$, the angles β_3 and θ_c are equal, and that for $\nu = \frac{1}{\sqrt{2}}$, β_3 and β'_3 are equal to.

The function $g_4(\theta)$ relative to the fourth and last pole is expressed :

$$g_4(\theta) = K_4(\theta)\ \sin(\theta-\beta) = \frac{1-2\nu\cos\theta + \sqrt{1-4\nu\cos\theta}}{2F^2\cos^2\theta}\ \sin(\theta-\beta) \qquad (2,27)$$

for $(1 - 4\nu\cos\theta > 0)$ and admits as derivative :

$$g'_4(\theta) = \frac{2\nu^2\cos\beta}{F^2\left[1-4\nu\cos\theta-(1-2\nu\cos\theta)\sqrt{1-4\nu\cos\theta}\right]\cos\theta}\left[\frac{1}{2}\left(\frac{\cos(2\theta-\beta)}{\cos\beta} - 1\right)-\sqrt{1-4\nu\cos\theta}\right]$$

(2,28)

In the present case we still have to study only the zeros of the second factor, except in respect of the integration domain limits.

When θ tends to $-\pi/2$, $g'_4(\theta)$ has a limited expansion with respect to $(-\theta+ \pi/2)$. The latter is written in the vicinity of $-\pi/2$:

$$g'_4(\theta) = \frac{1}{F^2(\theta+\frac{\pi}{2})^3}\left[2\cos\beta + (\sin\beta-2\nu\cos\beta)(\theta+\frac{\pi}{2}) + o(\theta+\frac{\pi}{2})\right] \qquad (2,29)$$

When θ tends to β, $g'_4(\theta)$ is identified with $K_4(\beta)$ which is always strictly positive.

If ν is smaller than ν_c, a graphic study of the open interval $]-\frac{\pi}{2}, \beta[$ enables us to conclude (figure 5).

Figure 6.

If ν is greater than ν_c, the expansion of $g'_4(\theta)$ are limited to the vicinities of $-\theta_c$ and $+\theta_c$:

$$g'_4(\theta) = 4\omega^2 \left[1 - \frac{tg\theta_c}{2\sqrt{\nu\sin\theta_c}} \sqrt{-\theta_c-\theta}\right] + o(\sqrt{-\theta_c-\theta}) \qquad (2,30)$$

$$g'_4(\theta) = 4\omega^2 \left[1 + \frac{tg\theta_c}{2\sqrt{\nu\sin\theta_c}} \sqrt{\theta-\theta_c}\right] + o(\sqrt{\theta-\theta_c}) \qquad (2,31)$$

It is clear that in these two cases $g'_4(\theta)$ has a non zero limits whatever ν and β may be.

Figure 7 enables us to illustrate the possible existence of zeros of $g'_4(\theta)$.

Figure 7

The results obtained in this way are shown together in figure 8. It should be noted that for $\nu < \nu_c$, there exists an angle β_4 defined by the existence of θ_4 in such a way that $g'_4(\theta_4) = g''_4(\theta_4) = 0$ is always smaller than β_K, and tends to zero when ν tends to ν_c. When $\nu > \nu_c$ there is no longer an angle β_4 corresponding to a double solution.

Figure 8,

2.4 Description of wave far field :

The respective contributions of each pole in relation to the parameters ν and β being determined, there remains the definition of the wave far field aspect according to the values of ν.

We recall for this purpose the general form obtained for the integral I (R). When, on the interval (a,b), the function g (θ) possesses maxima at the points $\theta = \alpha_M$, minima at the points $\theta = \alpha_m$ and horizontal tangent inflection points at the points $\theta = \alpha_i$, then :

$$I(R) = \sum_{\alpha_M} f(\alpha_M, R)\sqrt{\frac{2\pi}{|g''(\alpha_M)|}} \; e^{i\left[Rg(\alpha_M) - \frac{\pi}{4}\right]} \; R^{-1/2}\left[1 + O(R^{-1/2})\right]$$

$$+ \sum_{\alpha_m} f(\alpha_m, R)\sqrt{\frac{2}{|g''(\alpha_m)|}} \; e^{i\left[Rg(\alpha_m) + \frac{\pi}{4}\right]} \; R^{-1/2}\left[1 + O(R^{-1/2})\right]$$

$$+ \sum_{\alpha_i} f(\alpha_i, R) \; \frac{\Gamma(1/3)}{\sqrt{3}}\left[\frac{6}{|g'''(\alpha_i)|}\right]^{1/3} e^{iRg(\alpha_i)} \; R^{-1/3}\left[1 + O(R^{-1/3})\right]$$

(2,32)

If one of the stationary points $(g'(\alpha) = 0 \; ; \; g''(\alpha) \neq 0)$ coincides with one of the integration limits, its contribution is equal to half the value given by (2.32). On the other hand, if, at one of the integration limits, we have $g'(\alpha) = 0$, $g'(\alpha) = 0$ and $g'''(\alpha) \neq 0$ the contribution of the integral differs both by its coefficient an by its phase. We obtain :

$$f(a,R)\,\frac{\Gamma(1/3)}{3}\left[\frac{6}{|g'''(a)|}\right]^{1/3} e^{i\left[Rg(a) \pm \frac{\pi}{6}\right]} \; R^{-1/3}\left[1 + O(R^{-1/3})\right] \qquad (2,33)$$

$$f(b,R)\,\frac{\Gamma(1/3)}{3}\left[\frac{6}{|g'''(b)|}\right]^{1/3} e^{i\left[Rg(b) \pm \frac{\pi}{6}\right]} \; R^{-1/3}\left[1 + O(R^{-1/3})\right] \qquad (3,34)$$

The sign + corresponds to the cases where $g'''(a) > 0$ and $g'''(b) < 0$, the sign - to the opposite cases.

If θ_i is the solution of the equation $g'_i(\theta) = \upsilon$ the wave observed in the direction β propagates according to the axis defined by $\theta_i (+\pi)$ and, in the absolute reference, has a celerity of :

$$\mathcal{C}_i = U \cos\theta_i \left[\frac{\tilde{\omega}}{FK_i(\theta_i)\cos\theta_i} - 1 \right] = \frac{g}{2\omega} \left[1 \pm \sqrt{1 \pm 4\nu\cos\theta_i} \right] \qquad (2,35)$$

Figure 9 gives the general aspect of the different contributions of the poles with respect to ν. On this subject, several remarks should be made.

When a double solution exists, it is obtained as being the limit of two distinct solutions, one of which is a maximum and the other a minimum. It follows that the two systems are in quadrature in the corresponding direction.

In the direction defined by $\beta_1 = \beta_3$, the tangent at the potential lines of one of the wave systems is parallel to the axis Ox. When ν is smaller than $1/\sqrt{2}$, this remark applies to the annular system, when ν is equal to $1/\sqrt{2}$ the two wave system having this property ; and when ν is greater than $1/\sqrt{2}$ it is the system having $\beta = \theta_c$ as asymptote which is concerned. A consequence of this property is that in the first case the annular system is more colosed than a half ring, whereas the contrary occurs in the second case.

When ν is equal to 1/4, the group velocity of the waves generated forwards by K_3 ont the axis Ox is equal to the speed U. In fact, the wave velocity on this axis is then :

$$\mathcal{C}_3 = \frac{g}{2\omega} = \frac{U}{2\nu} = 2U \qquad (2,36)$$

The celerity is equal to twice the groupe velocity. In the relative frame, the velocity of these waves is thus equal to U, and the group velocity is zero. Then we observe a phenomenon of energy cramming.

The formulae (2.35) show that in the absolute frame the celerities $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are always positive whereas $\mathcal{C}_2$ is always negative.

The identification of the curves obtained is not trivial. Indeed, let us remember that in the case of the Neumann Kelvin problem, the potential lines are involutes of astroids.

2,5 Radiation condition at infinity :

The asymptotic wave field analysis, and particularly the determination of the velocity of the different wave systems, provides us a simple formulation of the radiation condition at infinity.

In the relative frame, all wave systems move away from the body generating them.

Of course, this general condition encompasses the two special cases of Froude number zero diffraction radiation and the Neumann Kelvin problem.

In addition, it is not verified by the parasite mathematical solution obtained by the model of the perfect fluid without ε. One may be convinced simply by remarking that the additionnai solution obtained without ε is deduced from the appropriate solution by a simmetry about the axis Oy.

III THE DEPTH IS FINITE

When the depth is finite, the velocity potential function is written in the following form :

$$\phi(M;t) = \phi_0(M;t) + \phi_1(M;t) + \phi_2(M;t) \qquad (3,1)$$

with :

$$\phi_0(M;t) = -\frac{\Omega}{4\pi}\left[\frac{1}{|MM'|} + \frac{1}{|MM'_1|} - \frac{2}{\pi h}\,\mathrm{Re}\left\{\int_{-\pi/2}^{+\pi/2} d\theta \int_0^{\infty} \frac{chK(Z+1)\,chK(Z'+1)}{chK}\, e^{K[-1+i\Omega]}\, dK\right\}\right]\cos\omega$$

$$\times \cos\omega t \qquad (3,2)$$

$$\phi_1(M;t) = -\frac{\Omega}{4\pi}\lim_{\tilde{\varepsilon}\to 0+}\left[-\frac{1}{\pi h}\,\mathrm{Re}\left\{e^{i\omega t}\int_{-\pi/2}^{+\pi/2} d\theta \int_0^{\infty} \frac{chK(Z+1)\,chK(Z'+1)\; e^{iK\Omega}\; K\, dK}{ck^2K\left[(\tilde{\omega}-FK\cos\theta)^2 - KthK - 2i\tilde{\varepsilon}(\tilde{\omega}-FK\cos\theta)\right]}\right\}\right]$$

$$(3,3)$$

$$\phi_2(M;t) = -\frac{\Omega}{4\pi}\lim_{\tilde{\varepsilon}\to 0+}\left[-\frac{1}{\pi h}\,\mathrm{Re}\left\{e^{-i\omega t}\int_{-\pi/2}^{+\pi/2} d\theta \int_0^{\infty} \frac{chK(Z+1)\,chK(Z'+1)\; e^{iK\Omega}\; K\, dK}{ch^2K\left[(\tilde{\omega}+FK\cos\theta)^2 - KthK + 2i\tilde{\varepsilon}(\tilde{\omega}+FK\cos\theta)\right]}\right\}\right]$$

$$(3,4)$$

Expressions in which M'_1 denotes the symmetry of M' with regard to the bottom, and where the sizes characteristic of flow are adimensionalised in relation to the depth h (1 is replaced by h in the formuleas (2.5.)). The denomination of the poles in paragraph 2 is conserved (K_1 and K_2 for $\phi_1(M;t)$, K_3 and K^ for $\phi_2(M;t)$ together with their order.

The integration of $\phi(M;t)$ in relation to the variable K is much more complicated than in the case of unlimited depth. We shall restrict ourselves to giving only the results necessary to the continuation of our study.

Regarding $\phi_0(M;t)$, the transformation of the product of the hyperbolical cosines into a sum of exponentials, then the serial expansion of the function $ch^{-1}K$ allow us to integrate with regard to K and .

$$\phi_0(M;t) = -\frac{Q}{4\pi}\left[\frac{1}{|MM'|} + \frac{1}{|MM'_1|} - \frac{1}{2\pi h}\sum_{n=0}^{\infty}(-1)^{n+1}\left\{\frac{1}{|MM'_{n1}|} + \frac{1}{|MM'_{n2}|} + \frac{1}{|MM'_{n3}|} + \frac{1}{|MM'_{n4}|}\right\}\right] \cos \quad (3,5)$$

with :

$$Z'_{n1} = +Z' + 2(n+1)$$

$$Z'_{n2} = +Z' - 2(n+1) \quad (3.6)$$

$$Z'_{n3} = -Z' + 2n$$

$$Z'_{n4} = -Z' - 2(n+2)$$

Regarding $\phi_1(M;t)$ and $\phi_2(M;t)$, we shall isolate the poles of the integrand, being careful to conserve their behaviour in their proximities and at infinity by writing :

$$\frac{K}{ch^2K\left[(\tilde{\omega}-FK\cos\theta)^2-KthK\right]} = P_1(K) + \frac{A(K_1)\ e^{-2K}}{K - K_1 + i\varepsilon'} + \frac{A(K_2)\ e^{-2K}}{K - K_2 + i\varepsilon'} \qquad (3,7)$$

$$\frac{K}{ch^2K\left[(\tilde{\omega}+FK\cos\theta)^2-KthK\right]} = P_2(K) + \frac{B(K_3)\ e^{-2K}}{K - K_3 - i\varepsilon''} + \frac{B(K_4)\ e^{-2K}}{K - K_4 + i\varepsilon''} \qquad (3,8)$$

with :

$$P_1(K) = \frac{K}{ch^2K\left[(\tilde{\omega}-FK\cos\theta)^2-KthK\right]} - \frac{A(K_1)\ e^{-2K}}{K - K_1} - \frac{A(K_2)\ e^{-2K}}{K - K_2} \qquad (3,9)$$

$$P_2(K) = \frac{K}{ch^2K\left[(\tilde{\omega}+FK\cos\theta)^2-KthK\right]} - \frac{B(K_3)\ e^{-2K}}{K - K_3} - \frac{B(K_4)\ e^{-2K}}{K - K_4} \qquad (3,10)$$

$$A(K) = \frac{K(1+thK)^2}{\frac{d}{dK}\left[(\tilde{\omega}-FK\cos\theta)^2-KthK\right]} \qquad (3,11)$$

$$B(K) = \frac{K(1+thK)^2}{\frac{d}{dK}\left[(\tilde{\omega}+FK\cos\theta)^2-KthK\right]} \qquad (3,12)$$

ε' and ε'' have the same behaviour as $\tilde{\varepsilon}$., It is then possible to integrate analytically the contribution of the poles after decomposing the product chK (Z + 1) ch K (Z' + 1) into a sum of four exponentials. As for the contribution of the regular part, it can be approximated by a series of exponential functions, which leads to a behaviour analogous to that of $\phi_0(M;t)$.

3,1 Near potential and far potential

As in paragraph 2,1, v/e can separate the terms which decrease a priori such as R^{-1} from the others. For the same reasons, only the real poles will be studied. By writing :

$$f_i(\theta) = A_i(K_i)\ chK_i(Z+1)\ chK_i(Z'+1)\, e^{-2K_i} \qquad i = 1\ ,\ 2 \qquad (3,13$$

$$f_i(\theta) = B_i(K_i)\ ckK_i(Z+1)\ chK_i(Z'+1)\, e^{-2K_i} \qquad i = 3\ ,\ 4 \qquad (3,14$$

we obtain :

$$\phi_{11}(M;t) = -\frac{\Omega}{2\pi h}\ \mathrm{Re}\left\{ie^{i\omega t}\left[\int_{-\pi/2}^{\beta} f_1(\theta)\ e^{iK_1\Omega}\ d\theta + \int_{-\pi/2}^{\beta} f_2(\theta)\ e^{iK_2\Omega}\ d\theta\right]\right\} \forall\ \nu \qquad (3,15$$

$$\phi_{21}(M;t) = +\frac{\Omega}{2\pi h}\ \mathrm{Re}\left\{ie^{-i\omega t}\left[\int_{\beta}^{+\pi/2} f_3(\theta)\ e^{iK_3\Omega}\ d\theta - \int_{-\pi/2}^{\beta} f_4(\theta)\ e^{iK_4\Omega}\ d\theta\right]\right\} \nu<\nu_c \qquad (3,16$$

$$\phi_{21}(M;t) = +\frac{\Omega}{2\pi h}\ \mathrm{Re}\left\{ie^{-i\omega t}\left[\int_{\beta}^{-\theta_c} f_3(\theta)\ e^{iK_3\Omega}\ d\theta + \int_{+\theta_c}^{+\pi/2} f_3(\theta)\ e^{iK_3\Omega}\ d\theta \right.\right.$$

$$\left.\left. - \int_{-\pi/2}^{\beta} f_4(\theta)\ e^{iK_4\Omega}\ d\theta\right]\right\} \qquad \nu>\nu_c \text{ and } -\frac{\pi}{2}<\beta<-\theta_c \qquad (3,17$$

$$\phi_{21}(M;t) = \frac{\Omega}{2\pi h} \text{ Re}\left\{ ie^{-i\omega t} \left[\int_{\theta_c}^{+\pi/2} f_3(\theta)\, e^{iK_3\Omega} d\theta - \int_{-\pi/2}^{-\theta_c} f_4(\theta)\, e^{iK_4\Omega} d\theta \right] \right\} \quad \nu > \nu_c \text{ and } |\beta| < |\theta_c| \qquad (3,18)$$

$$\phi_{21}(M;t) = \frac{\Omega}{2\pi h} \text{ Re}\left\{ ie^{-i\omega t} \left[\int_{\beta}^{+\pi/2} f_3(\theta)\, e^{iK_3\Omega} d\theta - \int_{-\pi/2}^{-\theta_c} f_4(\theta)\, e^{iK_4\Omega} d\theta \right.\right.$$

$$\left.\left. - \int_{\theta_c}^{\beta} f_4(\theta)\, e^{iK_4\Omega} d\theta \right] \right\} \quad \nu > \nu_c \text{ and } \theta_c < \beta < \frac{\pi}{2} \qquad (3,19)$$

The far potential is thus written again in a form favorable to the application of the stationary phase principle. Nevertheless, a new difficulty arises here in that the poles are no longer solutions of algebraic equations and that it is not nossible to put them in an analytical form. We shall remouve this difficulty by seeking analytical approximations having the same basic characters as the exact solutions. We shall then obtain information which will no longer be quantitative, but which will retain a qualitative significance in that the results will be qualitatively independent of the approximation found.

3,2 Analytical approximation of the poles :

We can express the poles simply only if they are solutions of algebraic equations the degree of which is equal to two at the most. We shall therefore approximate $\sqrt{K\tanh K}$ by a function of the following form :

$$y\,(K) = \frac{aK^2 + bK + c}{K + d} \qquad (3.20)$$

since the equations to be solved are :

$$\sqrt{K \text{ th } K} = \pm (\tilde{\omega} \pm F K \cos\theta) \qquad (3,21)$$

By imposing the value of the function and of its derivative at the origin, y (K) becomes :

$$y(K) = \frac{aK^2 + bK}{K + b} \qquad (3,22)$$

The parameters a and are determined by imposing on y(K) to connect tangentially with $\sqrt{K}$ in an abscissa point K_0. We then obtain for K_0 greater than the unit and different from 9/4 :

$$a = -\frac{1}{K_0}\left[\frac{2\sqrt{K_0} - K_0}{2\sqrt{K_0} - 3}\right] \qquad (3,23)$$

$$b = \frac{K_0}{2\sqrt{K_0} - 3} \qquad (3,24)$$

For K greater than K_0, $\sqrt{K \text{ th } K}$ is approximated by $\sqrt{K}$. We then find the solution in unlimited depth.

The figure 11 represents the approximation obtained for $K_0 = 2.2$, a qualitatively acceptable approximation since the characterisatics of $\sqrt{K \text{ th } K}$ are conserved, and particularly in the proximity of origin.

The approximate poles are solutions of the following equations :

$$\tilde{\omega} \pm FK\cos\theta = \pm \frac{aK^2 + bK}{K + b} \qquad (3,25)$$

It is clear that only the solutions coinciding with those obtained in ulimited depth at the connection point K_0, would be suitable. These solutions are expressed on the interval [o , K_0], and for real K_3 and K_4 •

$$K_1 = -\frac{b(1+F\cos\theta)-\tilde{\omega}+\sqrt{[b(1+F\cos\theta)-\tilde{\omega}]^2+4(a+F\cos\theta)b\tilde{\omega}}}{2(a+F\cos\theta)} \quad (3,26)$$

$$K_2 = -\frac{b(1-F\cos\theta)+\tilde{\omega}-\sqrt{[b(1-F\cos\theta)+\tilde{\omega}]^2-4(a-F\cos\theta)b\tilde{\omega}}}{2(a-F\cos\theta)} \quad (3,27)$$

$$K_3 = -\frac{b(1-F\cos\theta)-\tilde{\omega}+\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}{2(a-F\cos\theta)} \quad (3,28)$$

$$K_4 = -\frac{b(1-F\cos\theta)-\tilde{\omega}-\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}{2(a-F\cos\theta)} \quad (3,29)$$

It is unfortunately no longer possible to write the poles in a form exclusively dependent on the two single parameters ν and β since $\tilde{\omega}$ and F intervene separately. We are therefore led to discuss the existence of extrema of the functions $g_i(\theta)$ according to the values of $\tilde{\omega}$,F and β.

3,3 Asymptotic expansion of $\phi_{11}(M;t)$

The function $g_1(\theta)$ is expressed :

$$g_1(\theta) = K_1(\theta)\sin(\theta - \beta) \quad (3,30)$$

and admits as derivative with regard to θ :

$$g_1'(\theta) = K_1 \left\{ \cos(\theta-\beta) - \sin(\theta-\beta) \frac{bF\sin\theta}{\sqrt{[b(1+F\cos\theta)-\tilde{\omega}]^2 + 4(a+F\cos\theta)b\tilde{\omega}}} \left[1 + \frac{\tilde{\omega}}{K_1(a+F\cos\theta)} + \frac{\sqrt{[b(1+F\cos\theta)-\tilde{\omega}]^2 + 4(a+F\cos\theta)b\tilde{\omega}}}{b(a+F\cos\theta)} \right] \right\} \quad (3,31)$$

since K-^ is real and positive whatever may be θ so that $\cos\theta$ belongs to the interval $[\frac{\tilde{\omega} - \sqrt{K_o}}{K_o F}, 1]$, we are led to the study of the solutions of the following equation :

$$\cot g(\theta-\beta) = \frac{bF\sin\theta}{\sqrt{[b(1+F\cos\theta)-\tilde{\omega}]^2 + 4(a+F\cos\theta)b\tilde{\omega}}} \left[1 + \frac{\tilde{\omega}}{K_1(a+F\cos\theta)} + \frac{\sqrt{[b(1+F\cos\theta)-\tilde{\omega}]^2 + 4(a+F\cos\theta)b\tilde{\omega}}}{b(a+F\cos\theta)} \right] \quad (3,32)$$

Figure 12 shows that there exists a couple $(\tilde{\omega}, F)$ and therefore a Strouhal number ν''_c so that on the interval $[-\frac{\pi}{2}, \beta]$:

- for $\nu < \nu''_c$:
  $g'_1(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta_1[$ $(\beta_1 > 0)$
  $g'_1(\theta)$ has one zero, minimum of $g_1(\theta)$, if $\beta \in [\beta_1, +\frac{\pi}{2}]$
- for $\nu = \nu''_c$:
  $g'_1(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta_1[$ $(\beta_1 > 0)$
  $g'_1(\theta)$ has double zero if $\beta = \beta_1$
  $g'_1(\theta)$ has one zero, minimum of $g_1(\theta)$, if $\beta \in]\beta_1, +\frac{\pi}{2}]$
- for $\nu > \nu''_c$:
  $g'_1(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta'_1[$ $(\beta'_1 > 0)$
  $g'_1(\theta)$ has a double zero if $\beta = \beta_1$

$g'_1(\theta)$ has two distinct zeros, a minimum and a maximum of $g_1(\theta)$ if $\beta \in]\beta'_1, \beta_1]$

$g'_1(\theta)$ has one zero, minimum of $g_1(\theta)$, if $\beta \in]\beta_1, +\frac{\pi}{2}]$

We therefore find the same conditions for existence of extrema and horizontal tangent inflection as in the case of unlimited depth but the values of ν''_c, β_1 and β'_1 depend of course on h.

Figure 12

The function $g_2(\theta)$ is expressed :

$$g_2(\theta) = K_2(\theta) \sin(\theta - \beta) \tag{3,33}$$

in which $K_2(\theta)$ is defined by the formula (3,27) if $\cos\theta > \frac{\tilde{\omega} + \sqrt{K_0}}{K_0 F}$ and by the same formula as in paragraph 2.2 if $\cos\theta < \frac{\tilde{\omega} + \sqrt{K_0}}{K_0 F}$, It therefore admit as derivative according to the values of :

$$g'_2(\theta) = K_2 \left\{ \cos(\theta-\beta) - \sin(\theta-\beta) \frac{bF\sin\theta}{\sqrt{[b(1-F\cos\theta)+\tilde{\omega}]^2 - 4(a-F\cos\theta)b\tilde{\omega}}} \left[1 - \frac{\tilde{\omega}}{K_2(a-F\cos\theta)} - \frac{\sqrt{[b(1-F\cos\theta)+\tilde{\omega}]^2 - 4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)} \right] \right\} \tag{3,34}$$

$$g'_2(\theta) = K_2 \left\{ \cos(\theta-\beta) - \sin(\theta-\beta) \left[\frac{2\nu\sin\theta(1+\sqrt{1+4\nu\cos\theta})}{(1+2\nu\cos\theta)\sqrt{1+4\nu\cos\theta} + (1+4\nu\cos\theta)} - 2\mathrm{tg}\theta \right] \right\} \tag{3,35}$$

Since K_2 is real and positive whatever θ may be, we are led to the study of the solutions of one of these expressions, according to the values of :

$$\mathrm{cotg}(\theta-\beta) = \frac{bF\sin\theta}{\sqrt{[b(1-F\cos\theta)+\tilde{\omega}]^2-4(a-F\cos\theta)b\tilde{\omega}}}\left[1-\frac{\tilde{\omega}}{K_2(a-F\cos\theta)} - \frac{\sqrt{[b(1-F\cos\theta)+\tilde{\omega}]^2-4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)}\right] \qquad (3,36)$$

$$\mathrm{cotg}(\theta-\beta) = \frac{2\nu\sin\theta(1+\sqrt{1+4\nu\cos\theta})}{(1+2\nu\cos\theta)\sqrt{1+4\nu\cos\theta}+(1+4\nu\cos\theta)} - 2\mathrm{tg}\theta \qquad (3,37)$$

Figure 13 shows that there exists a positive angle, but smaller than that of paragraph 2, so that, for all values of   :

- $g'_2(\theta)$ is never cancelled if $\beta \in [-\frac{\pi}{2}, \beta_2[$
- $g'_2(\theta)$ has a double zero if $\beta = \beta_2$
- $g'_2(\theta)$ has two distinct zeros, a minimum and a maximum of $g_2(\theta)$ if $\beta \in]\beta_2, +\frac{\pi}{2}[$
- $g'_2(\theta)$ has one zero, maximum of $g_2(\theta)$ if $\beta = +\frac{\pi}{2}$.

We again find the same conditions as in the case of unlimited depth, β_2 being this time a function of h.

Figure 13

3,4 Asymptotic expansion of Φ_2 (M;t)

The function $g_3(\theta)$ isexpressed $\left(\frac{\sqrt{K_0}-\tilde{\omega}}{FK_0} > 1\right)$: :

$$g_3(\theta) = K_3(\theta)\sin(\theta-\beta) \tag{3,38}$$

and admits as derivative with regard to θ :

$$g_3'(\theta) = K_3\left\{\cos(\theta-\beta)+\sin(\theta-\beta)\frac{bF\sin\theta}{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}\left[1+\frac{\tilde{\omega}}{K_3(a-F\cos\theta)}+\frac{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)}\right]\right\} \tag{3,39}$$

since K_3 is real and positive whatever θ may be so that $\cos\theta$ belongs to the interval $\left[\frac{\sqrt{K_0}-\tilde{\omega}}{FK_0}, 1\right]$, we are then led to the study of the solutions of the following equation :

$$\operatorname{cotg}(\theta-\beta) = \frac{-bF\sin\theta}{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}\left[1+\frac{\tilde{\omega}}{K_3(a-F\cos\theta)}+\frac{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2+4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)}\right] \tag{3,40}$$

Figure 14 illustrates several modes according to the values of the couples $(\tilde{\omega},F)$:

- For $\nu < \nu_c$:

  $g'_3(\theta)$ has one zero, maximum of $g_3(\theta)$ if $\beta \in [-\frac{\pi}{2}, \beta_3]$ $(\beta_3 = \beta_1)$

  $g'_3(\theta)$ has no zero if $\beta \in]\beta_3, +\frac{\pi}{2}]$

- for $\nu_c < \nu < \nu'_c$:

  $g'_3(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta'_3[$

  $g'_3(\theta)$ has a double zero if $\beta = \beta'_3$

  $g'_3(\theta)$ has two distinct zeros, a minimum and a maximum of $g_3(\theta)$ if $\beta \in]\beta'_3, \theta_c[$

  $g'_3(\theta)$ has one zero, maximum of $g_3(\theta)$ if $\beta \in [\theta_c, \beta_3]$

  $g'_3(\theta)$ has no zero if $\beta \in]\beta_3, \frac{\pi}{2}]$

- for $\nu = \nu'_c$:

  $g'_3(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta'_3[$

  $g'_3(\theta)$ has a double zero if $\beta = \beta'_3$

  $g'_3(\theta)$ has two distinct zeros, a minimum and a maximum of $g_3(\theta)$ if $\beta \in]\beta'_3, \theta_c[$

  $g'_3(\theta)$ has one zero, maximum of $g_3(\theta)$ if $\beta = \theta_c = \beta_3 = \beta_1$

  $g'_3(\theta)$ is never cancelled if $\beta \in]\beta_3, +\frac{\pi}{2}]$

- for $\nu'_c < \nu < \nu''_c$:

  $g'_3(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta'_3[$

  $g'_3(\theta)$ has a double zero if $\beta = \beta'_3$

  $g'_3(\theta)$ has two distinct zeros, a minimum and a maximum of $g_3(\theta)$ if $\beta \in]\beta'_3, \beta_3]$

  $g'_3(\theta)$ has one zero, minimum of $g_3(\theta)$ if $\beta \in]\beta_3, \theta_c[$

  $g'_3(\theta)$ is never cancelled if $\beta \in [\theta_c, +\frac{\pi}{2}]$

- for $\nu = \nu''_c$:

  $g'_3(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta_3[$ $(\beta_3 = \beta'_3)$

  $g'_3(\theta)$ has a double zero if $\beta = \beta_3 = \beta'_3$

  $g'_3(\theta)$ has one zero, minimum of $g_3(\theta)$ if $\beta \in]\beta_3, \theta_c[$

  $g'_3(\theta)$ is never cancelled if $\beta \in [\theta_c, +\frac{\pi}{2}]$

- for $\nu > \nu''_c$:

  $g'_3(\theta)$ has no zero if $\beta \in [-\frac{\pi}{2}, \beta_3[$

  $g'_3(\theta)$ has one zero, minimum of $g_3(\theta)$ if $\beta \in [\beta_3, \theta_c[$

  $g'_3(\theta)$ is never cancelled if $\beta \in [\theta_c + \frac{\pi}{2}]$

Here too we find the same conditions as in the case of unlimited depth β_3, β'_3 and θ_c being functions of h.

Figure 14

The function $g_4(\theta)$ is expressed :

$$g_4(\theta) = K_4(\theta) \sin(\theta - \beta) \tag{3,41}$$

where $K_4(\theta)$ is defined by the formula (3.29) if $\cos\theta > \dfrac{\tilde{\omega} - \sqrt{Ko}}{F\ Ko}$, and by the same formula as in paragraph 2.3 if $\cos\theta < \dfrac{\tilde{\omega} - \sqrt{Ko}}{F\ Ko}$. It therefore admits as derivative according to the values of $\cos\theta$:

$$g'_4(\theta) = K_4 \left\{ \cos(\theta-\beta) - \sin(\theta-\beta) \frac{bF\cos\theta}{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2 + 4(a-F\cos\theta)b\tilde{\omega}}} \left[1 + \frac{\tilde{\omega}}{K_4(a-F\cos\theta)} - \frac{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2 + 4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)}\right]\right\} \tag{3,42}$$

$$g'_4(\theta) = K_4 \left\{ \cos(\theta-\beta) - \sin(\theta-\beta) \left[\frac{-2\nu\sin\theta(1+\sqrt{1-4\nu\cos\theta})}{(1-2\nu\cos\theta)\sqrt{1-4\nu\cos\theta}+(1-4\nu\cos\theta)} - 2\ \mathrm{tg}\,\theta\right]\right\} \tag{3,43}$$

since K_4^* is real and positive whatever θ may be, we are led to study of the solutions of one of theses expressions according to the values of $\cos\theta$:

$$\mathrm{cotg}(\theta-\beta) = \frac{bF\sin\theta}{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2 + 4(a-F\cos\theta)b\tilde{\omega}}} \left[1 + \frac{\tilde{\omega}}{K_4(a-F\cos\theta)} - \frac{\sqrt{[b(1-F\cos\theta)-\tilde{\omega}]^2 + 4(a-F\cos\theta)b\tilde{\omega}}}{b(a-F\cos\theta)}\right] \tag{3,44}$$

$$\cot g(\theta-\beta) = \frac{-2\nu\sin\theta(1+\sqrt{1-4\nu\cos\theta})}{(1-2\nu\cos\theta)\sqrt{1-4\nu\cos\theta}+(1-4\nu\cos\theta)} - 2tg\theta \qquad (3,45)$$

Figure 15 illustrates a couple ($\tilde{\omega}$, F) corresponding to ν_c so that :

- for $\nu < \nu_c$:

  $g'_4(\theta)$ admits no zero if $\beta \in [-\frac{\pi}{2}, \beta_4[$

  $g'_4(\theta)$ has a double zero if $\beta = \beta_4$

  $g'_4(\theta)$ has two zeros, a minimum and a maximum of $g_4(\theta)$ if $\beta \in]\beta_4, +\frac{\pi}{2}[$

  $g'_4(\theta)$ has one zero, maximum of $g'_4(\theta)$ if $\beta = +\frac{\pi}{2}$

- for $\nu > \nu_c$:

  $g'_4(\theta)$ is never cancelled if $\beta \in [-\frac{\pi}{2}, \theta_c]$

  $g'_4(\theta)$ has one zero, minimum of $g_4(\theta)$ if $\beta \in]\theta_c, +\frac{\pi}{2}[$

  $g'_4(\theta)$ has no zero if $\beta = +\frac{\pi}{2}$

Once again we find the same configuration as in the case of unlimited depth.

Figure 15

3,5 Generalisation :

For the four poles we have found the same results as those brought together on figure 8. However, before accepting them such as they are, several remarks are necessary.

If the figures proposed take into account only one approximation ($K_\theta = 2.2$), different values of K_θ have been studied and lead to the same results. In the case where $\tilde{\omega}$ is high, the poles $\underline{K}_1$ and $\underline{K}_3$ must be studied as K_2

and K_4, from the two formulations corresponding to the cases of finite and infinite depth according to the values of cos θ , but the results remain qualitatively valid.

It should be noted that the true poles lead to continuous curves the tangents of which are also continuous. It follows that the junctions between the arcs relative to finite depth on the one hand, and infinite depth on the other, must be observed critically, in order not to introduce errouneous results.

The most important remark concerns the particular value F = 1.0, which corresponds to a change of mode for the Neumann-Kelvin problem, but which does not introduce any radical change regarding the problem tackled here.

4 CONCLUSION

The study carried out has enabled us to give a complete description of the weve far field in the case of unlimited depth, and to show that for a finite depth the general aspect of the asymptotic wave field is qualitatively conserved, although the values of the different parameters depend on the depth.

In particular it follows that the radiation condition at infinity is expressed in the same way in both these cases.

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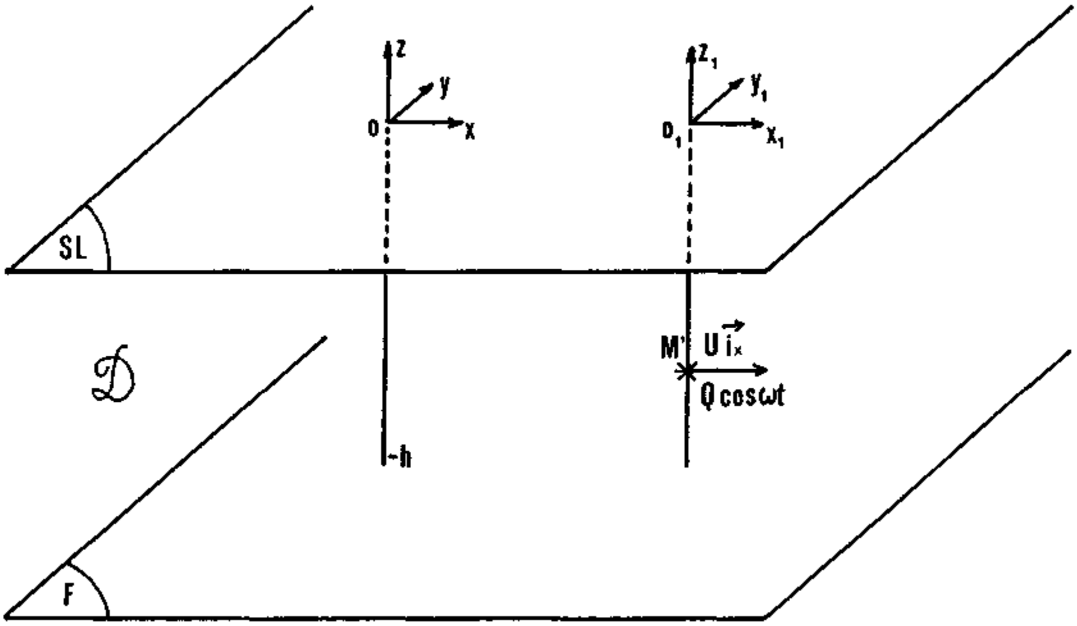
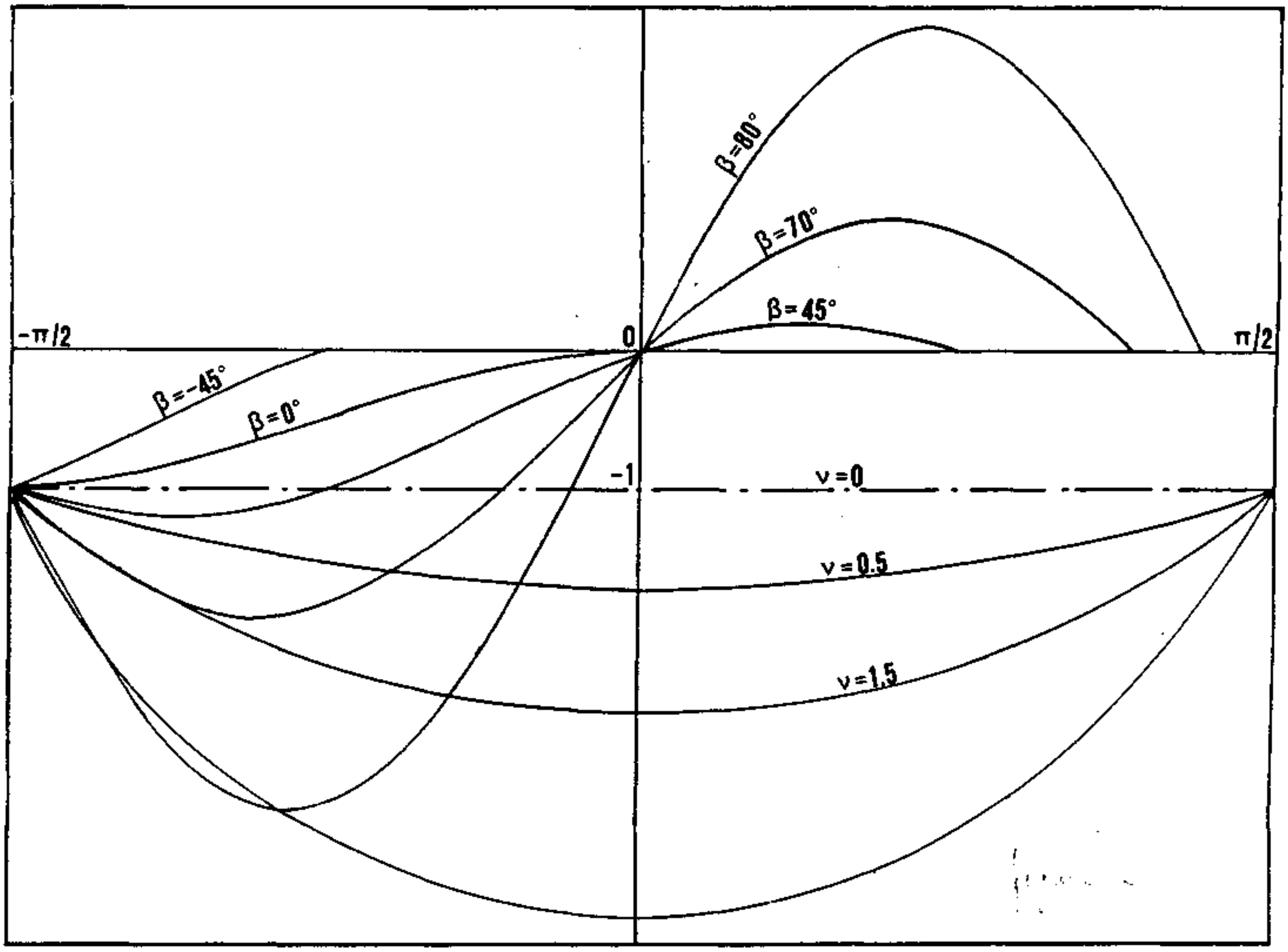


z
y
o
x
z1
y1
O1
x1
SL
M'
U i_x
Q cosωt
-h
F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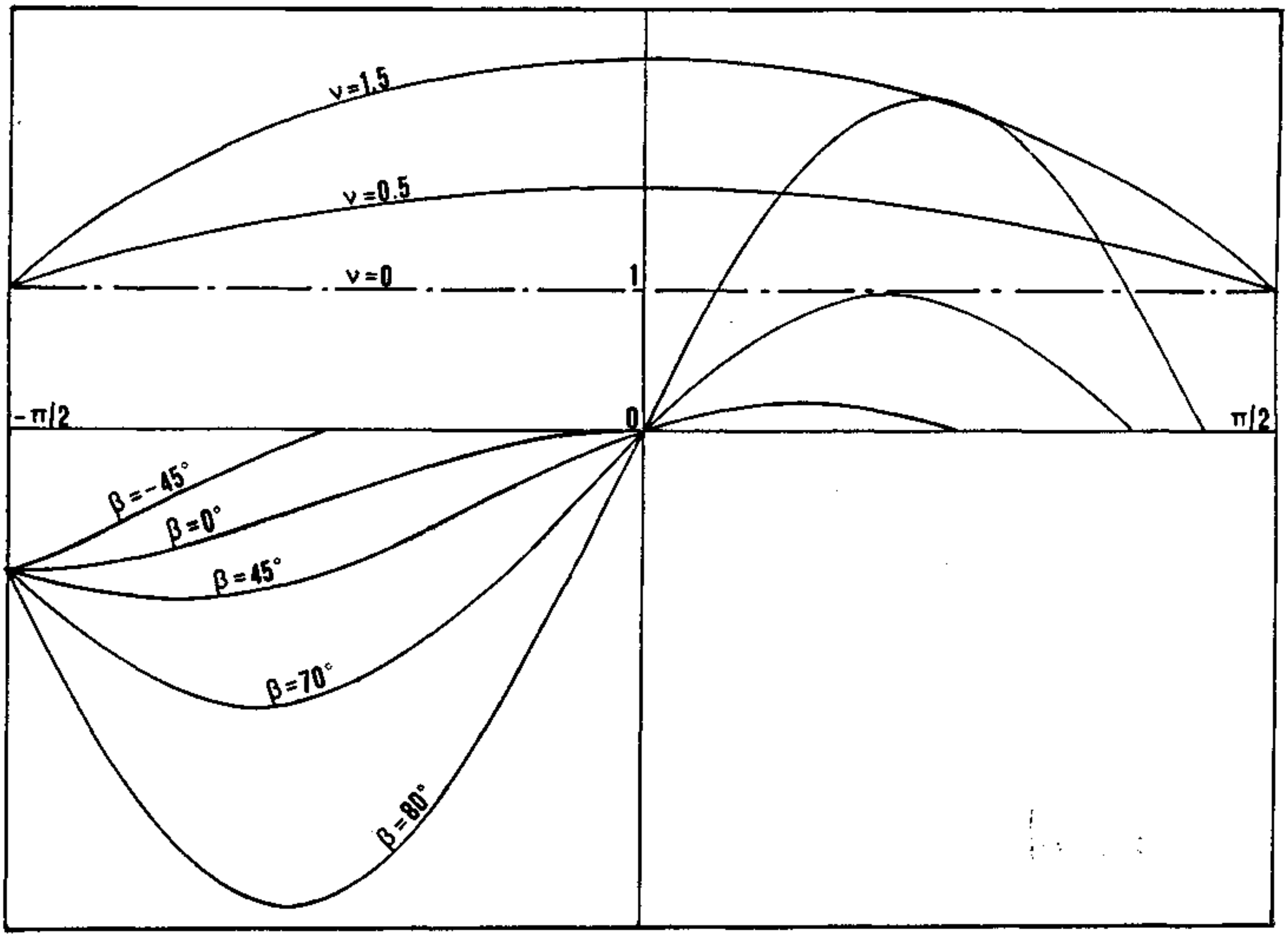
β=70°
β=45°
-π/2
0
π/2
β=-45°
β=0°
-1
ν=0
ν=0.5
ν=1.5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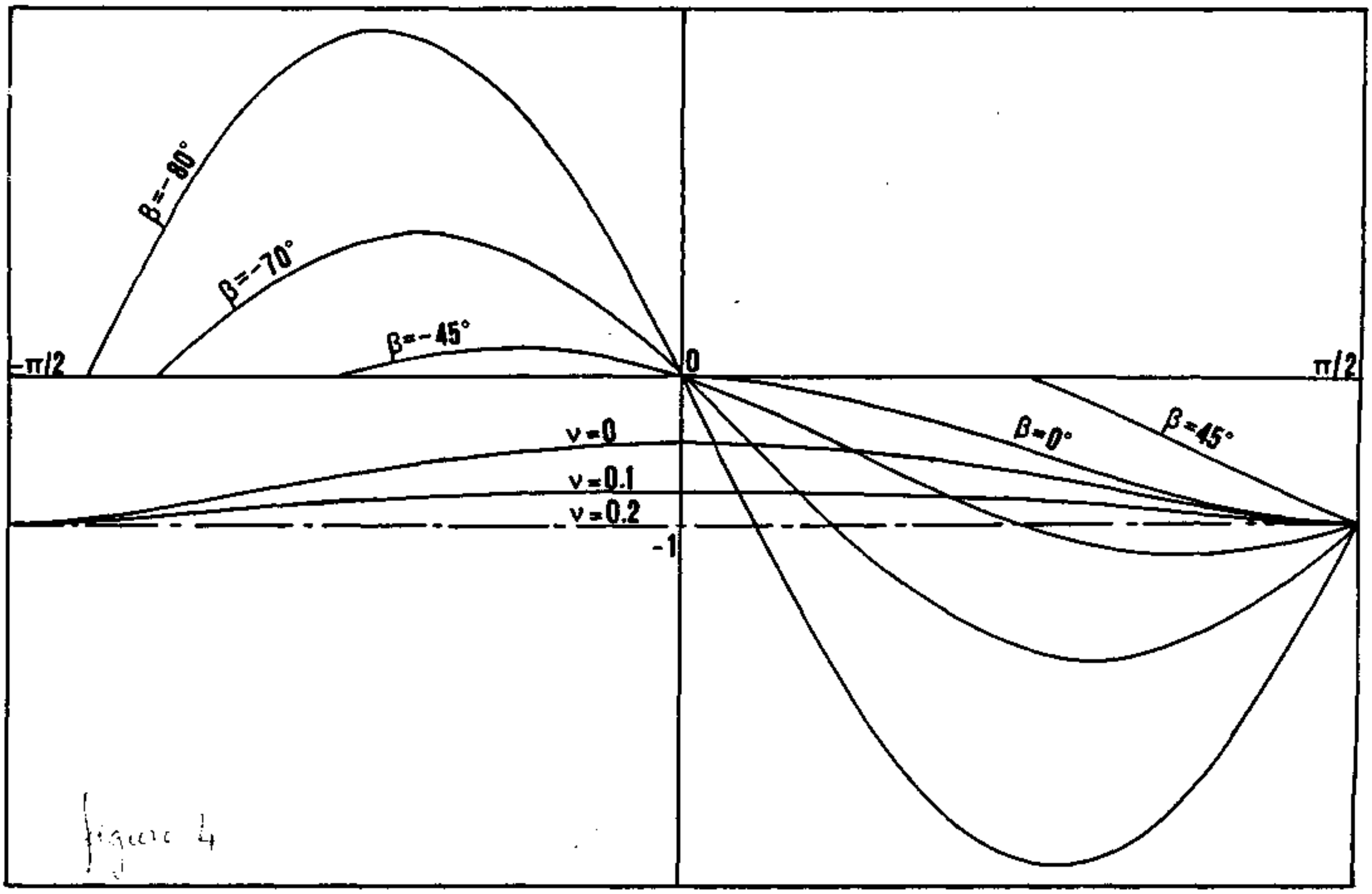
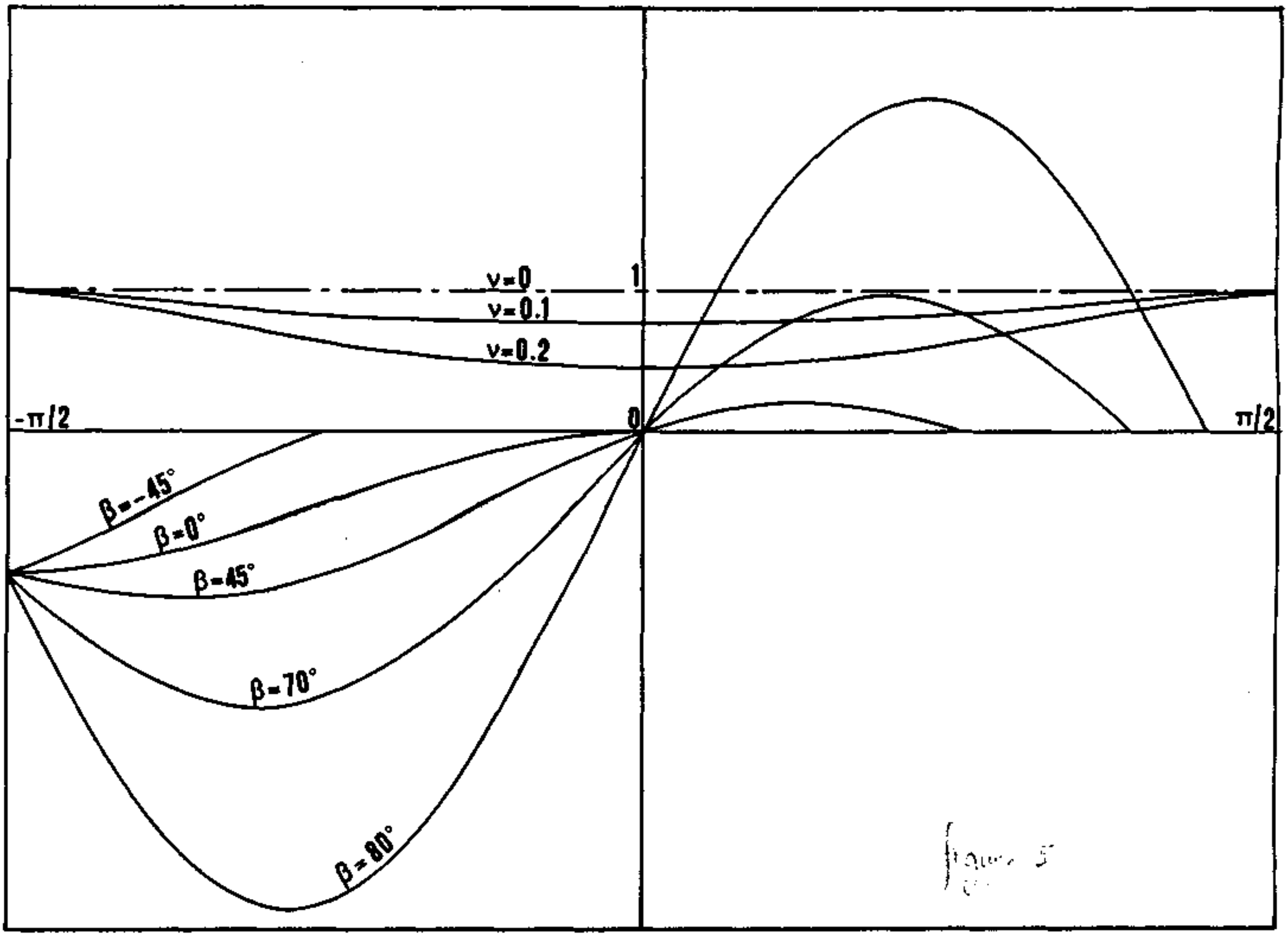
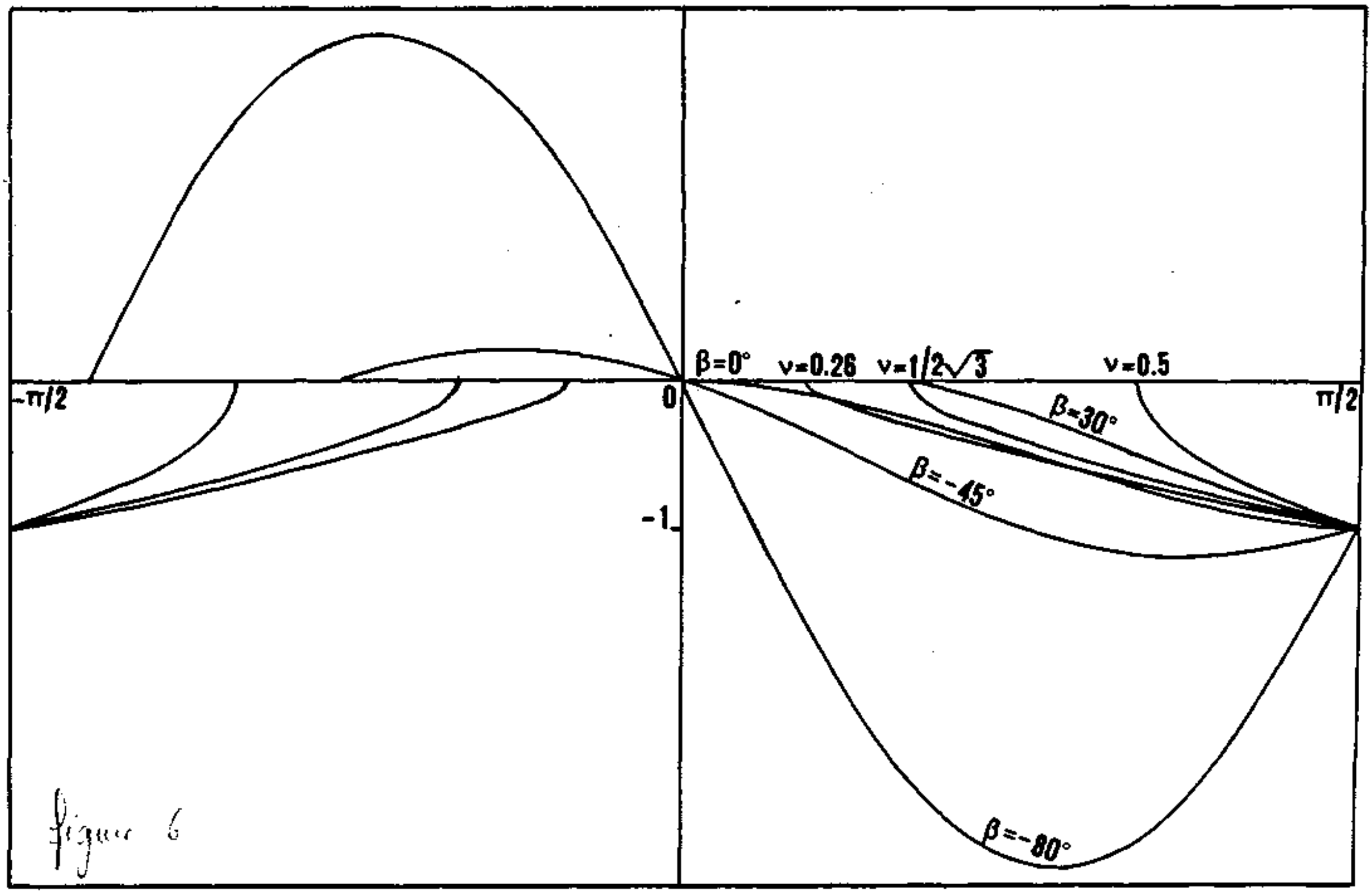
ν=0.5
ν=0
1
-π/2
0
π/2
β=-45°
β=0°
β=45°
β=70°
β=80°


β=-80°
β=-70°
β=-45°
-π/2
0
π/2
β=0°
β=45°
ν=0
ν=0.1
ν=0.2
-1
figure 4


ν=0
1
ν=0.1
ν=0.2
-π/2
0
π/2
β=-45°
β=0°
β=45°
β=70°
β=80°
figure 5


β=0°
ν=0.26
ν=1/2√3
ν=0.5
-π/2
0
π/2
β=30°
β=-45°
-1
β=-80°
figure 6

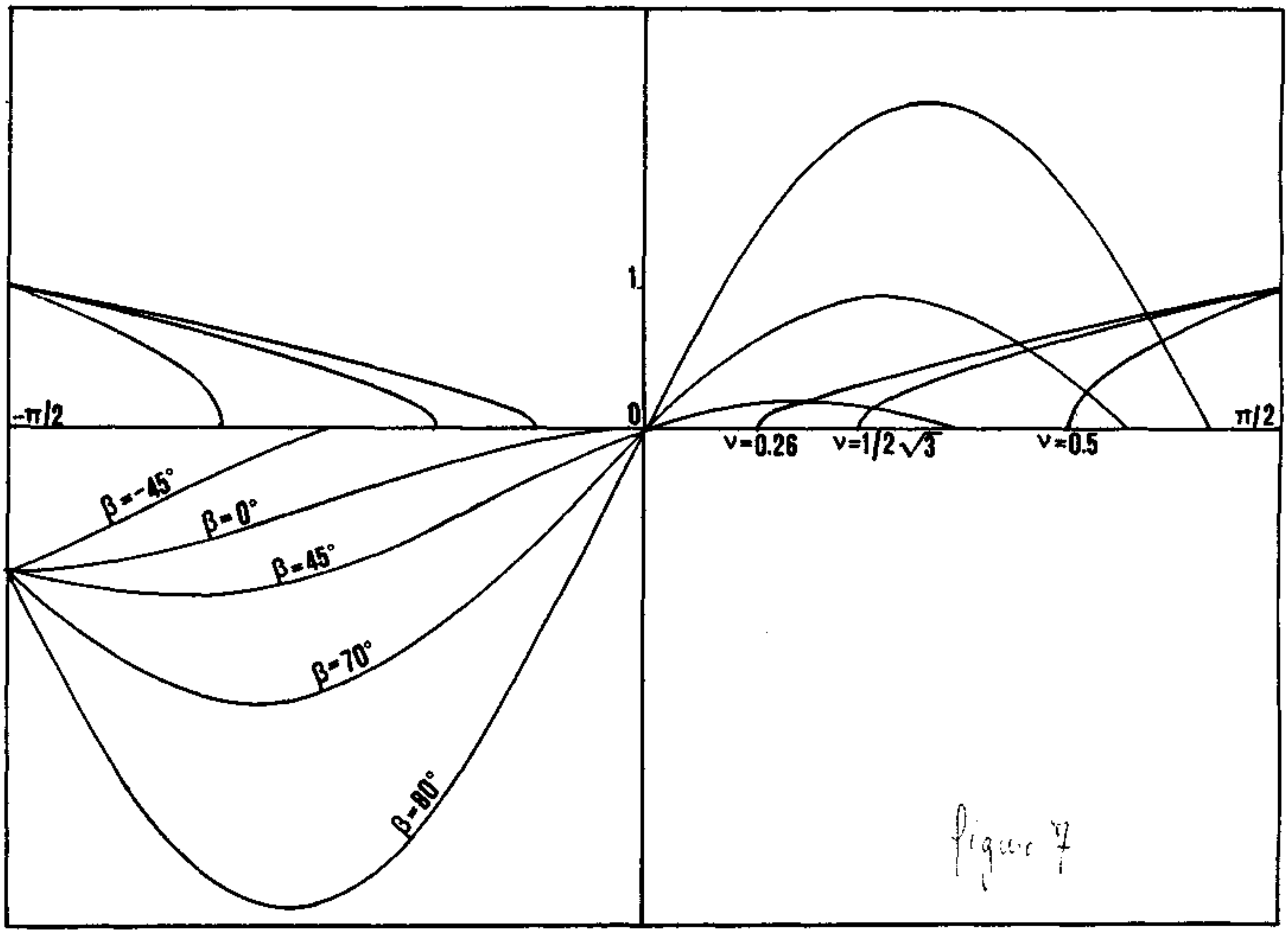
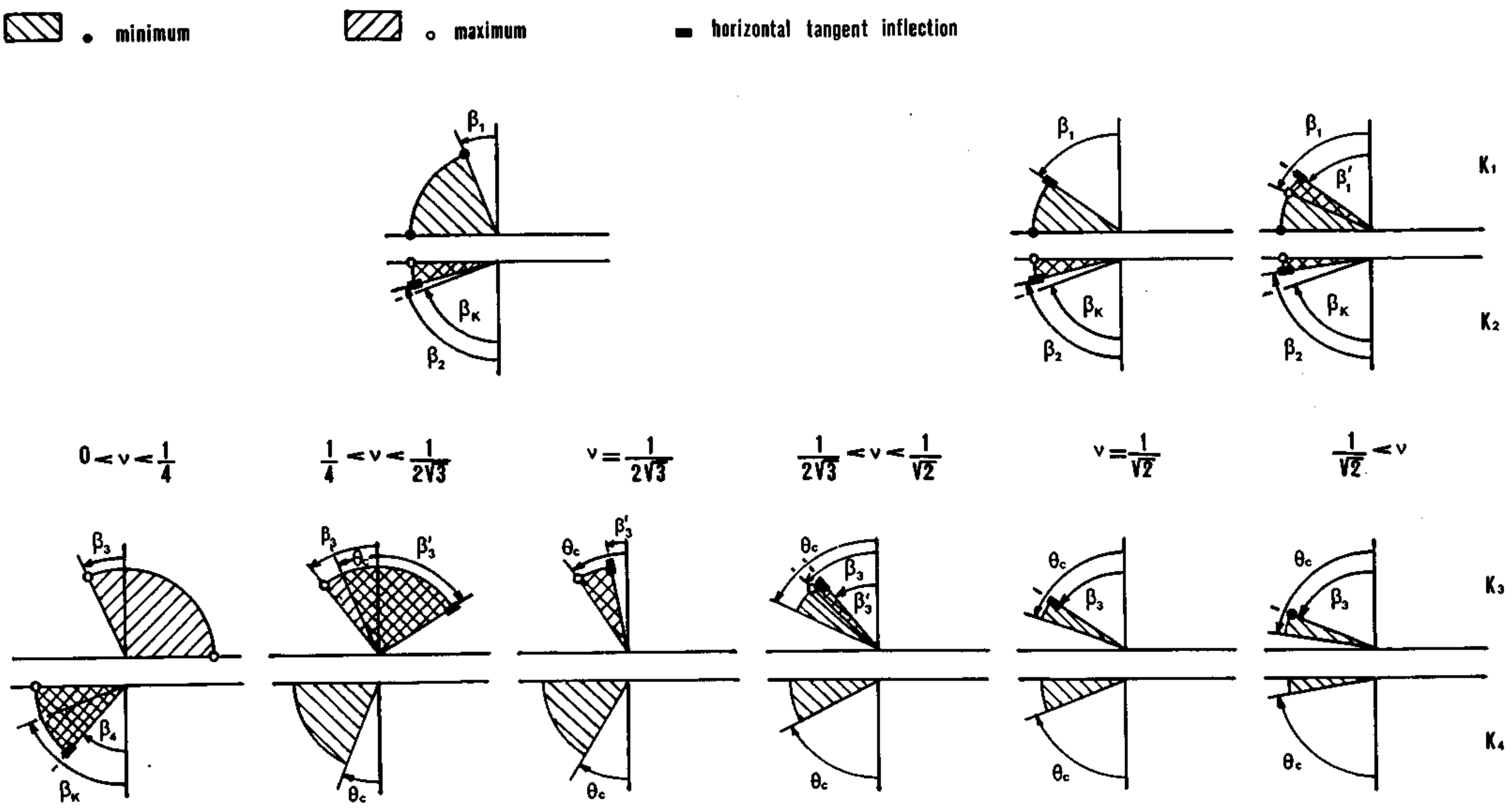

1
-π/2
0
π/2
ν=0.26
ν=1/2√3
ν=0.5
β=-45°
β=0°
β=45°
β=70°
β=80°
figure 7

figure 8



minimum
maximum
horizontal tangent inflection
β1
β1'
βK
β2
K1
K2
0 < ν < 1/4
1/4 < ν < 1/2√3
ν = 1/2√3
1/2√3 < ν < 1/√2
ν = 1/√2
1/√2 < ν
β3
β3'
β4
θc
K3
K4

figure 9

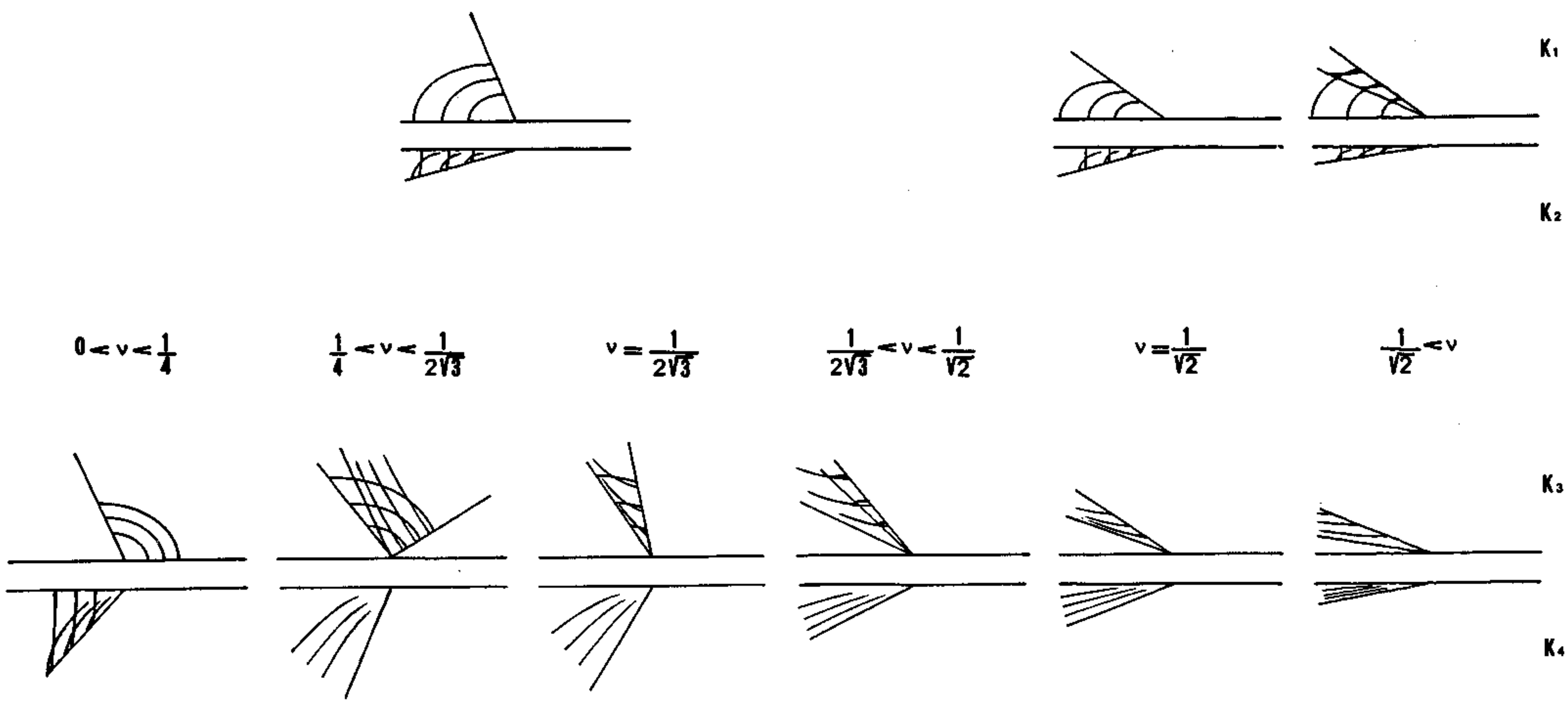
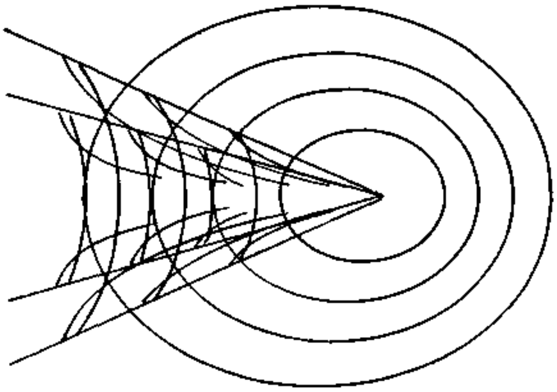
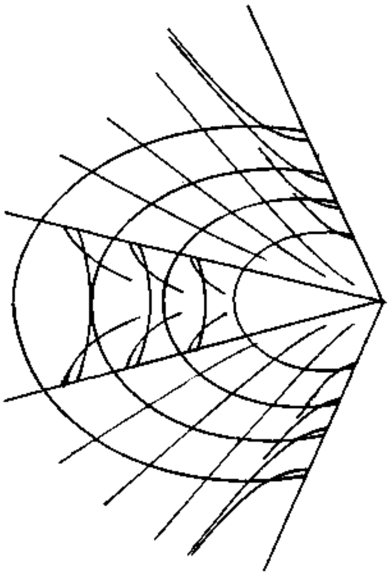

K1
K2
$0<\nu<\frac{1}{4}$
$\frac{1}{4}<\nu<\frac{1}{2\sqrt{3}}$
$\nu=\frac{1}{2\sqrt{3}}$
$\frac{1}{2\sqrt{3}}<\nu<\frac{1}{\sqrt{2}}$
$\nu=\frac{1}{\sqrt{2}}$
$\frac{1}{\sqrt{2}}<\nu$
K3
K4

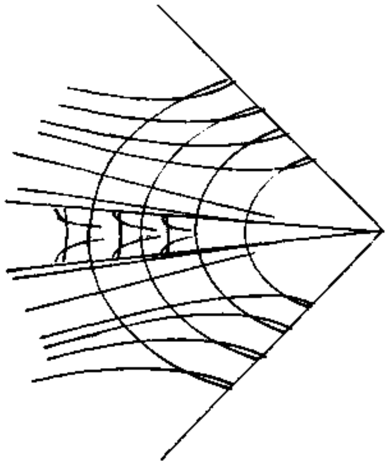
figure 10



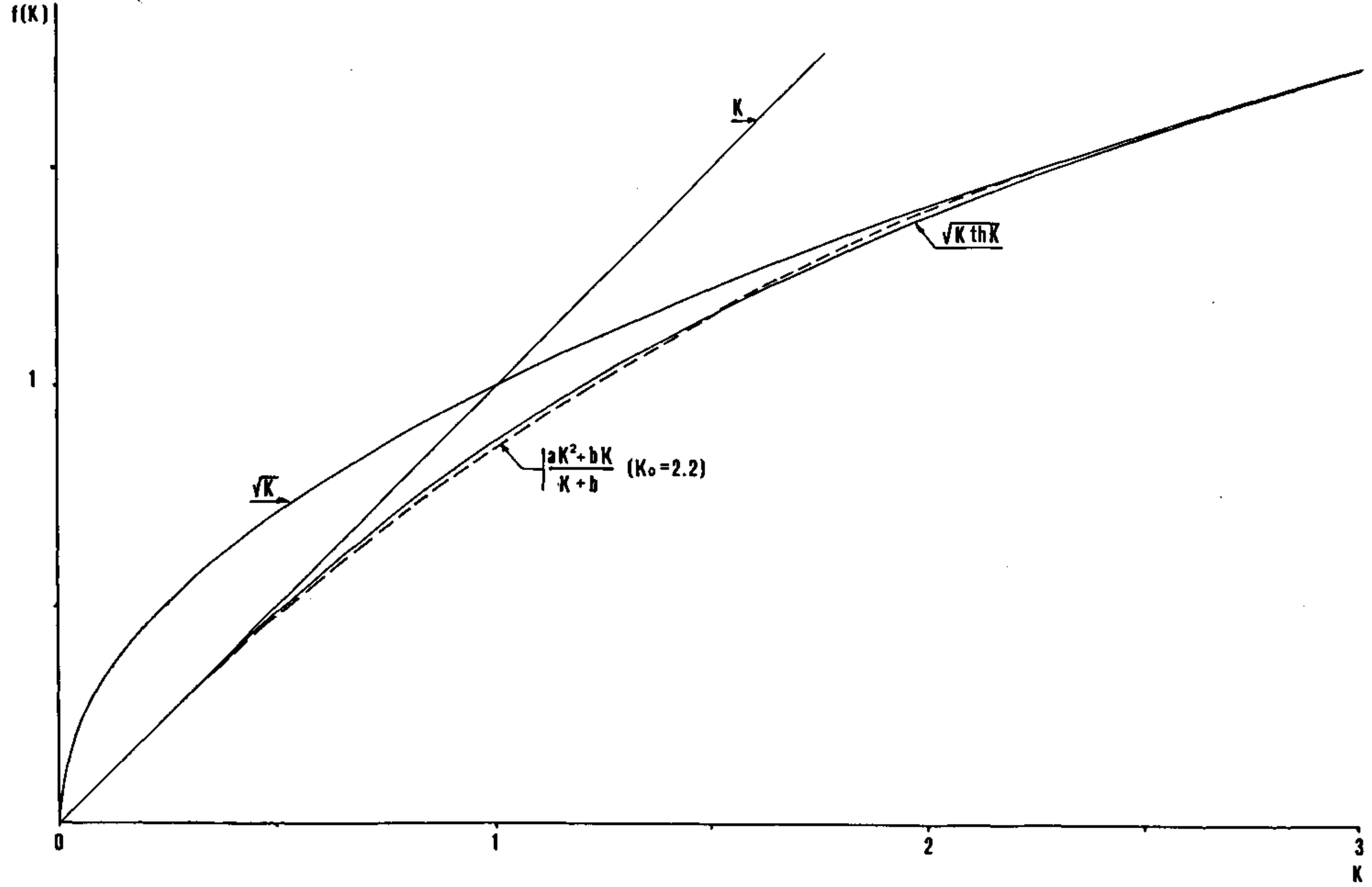
$\nu < \frac{1}{4}$



$\frac{1}{4} < \nu < \frac{1}{\sqrt{2}}$



$\frac{1}{\sqrt{2}} < \nu$


f(K)
1
0
1
2
3
K
K
√K
√K thK
(aK² + bK)/(K + b) (K₀ = 2.2)

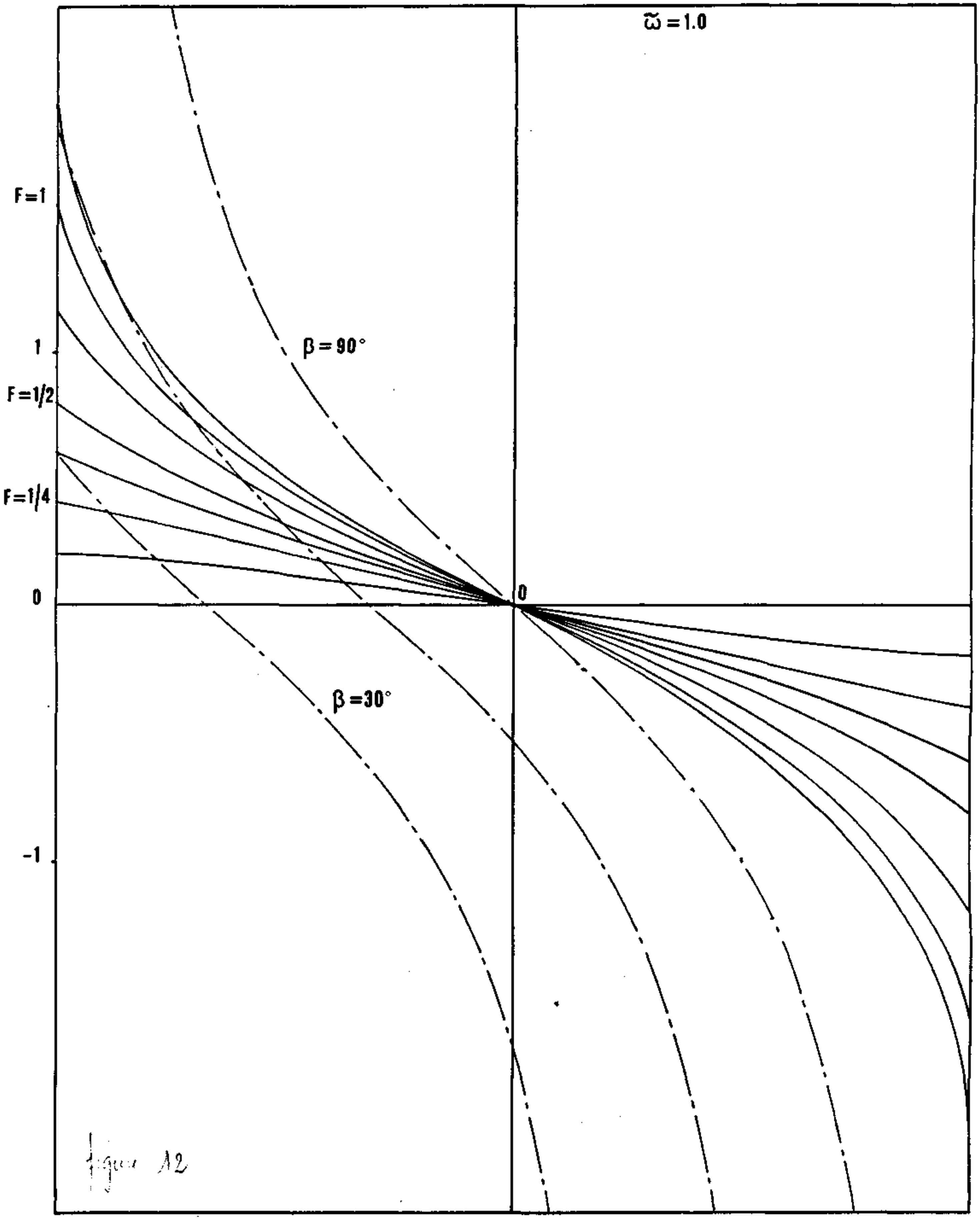


figure 12

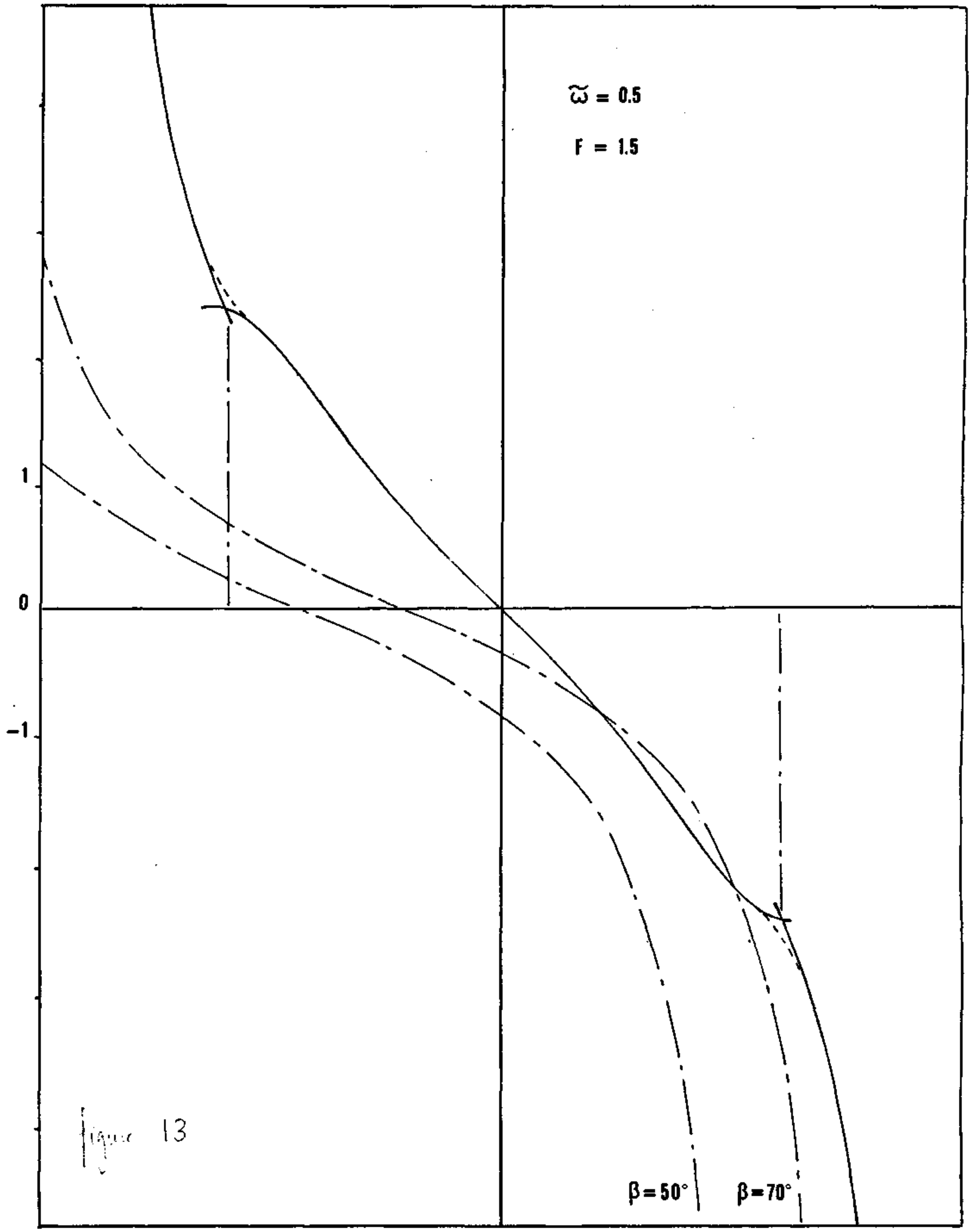
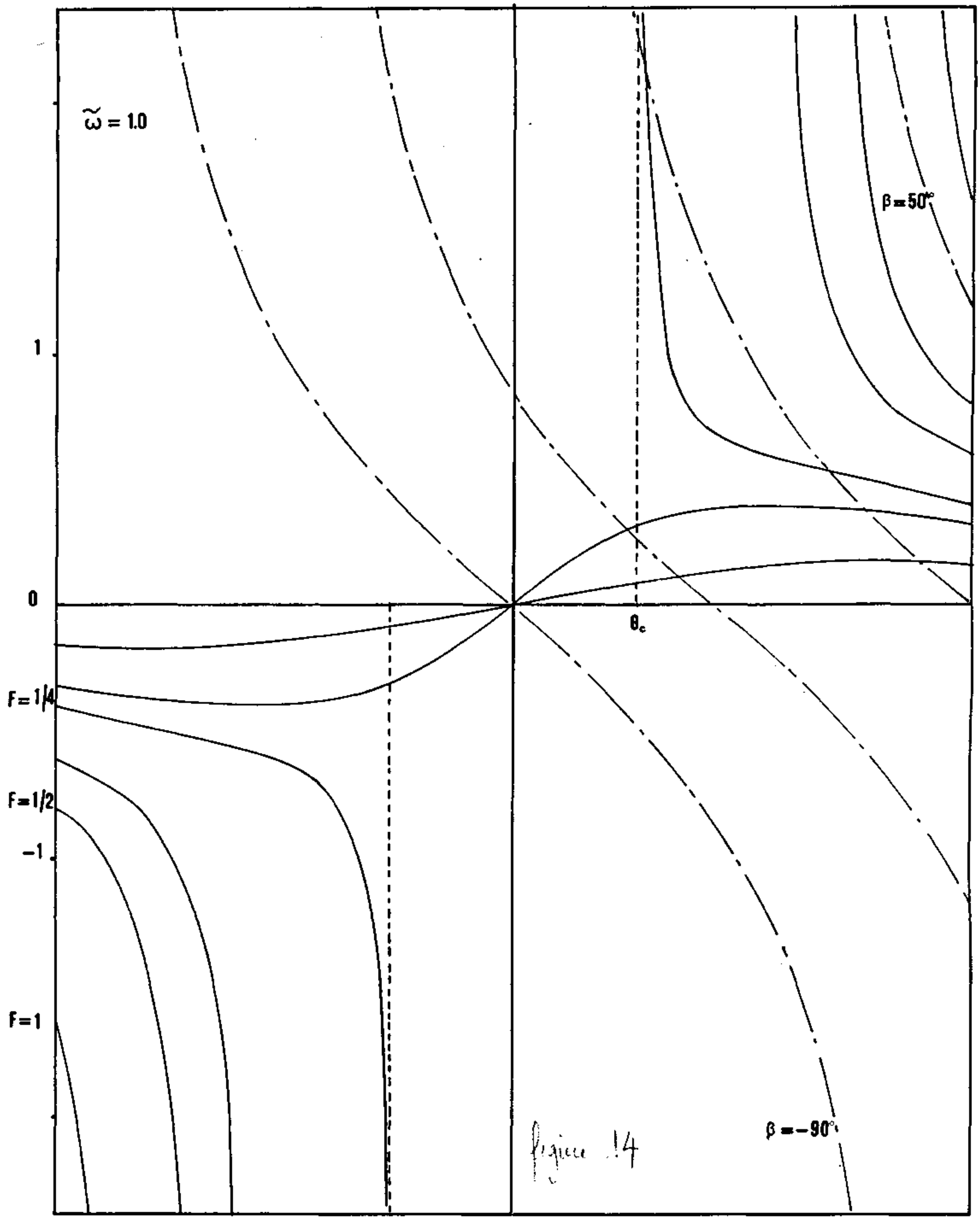
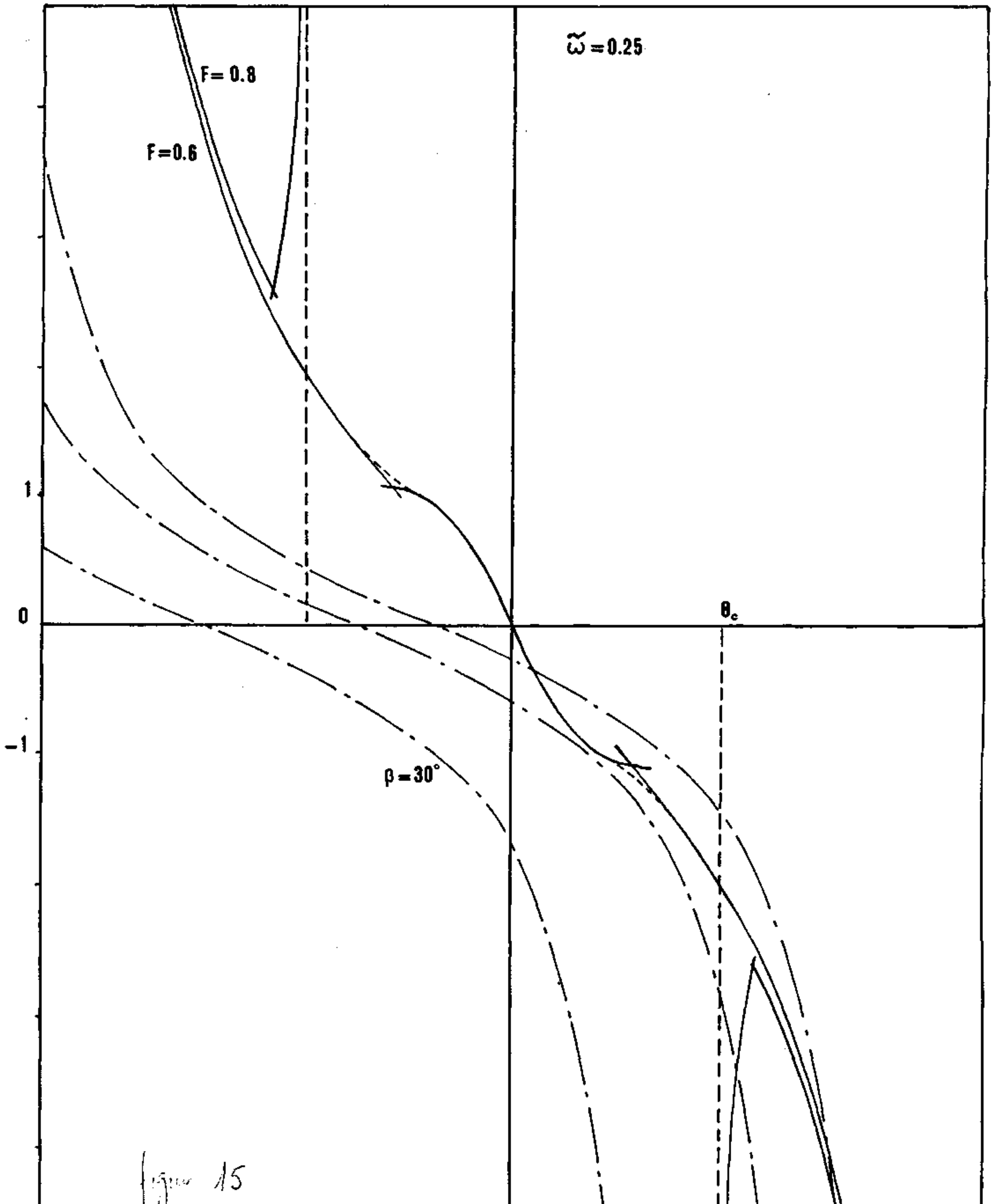


figure 13



$\tilde{\omega} = 1.0$
1
0
-1
F=1/4
F=1/2
F=1
θ_c
β=50°
β=−90°
figure 14



$\tilde{\omega} = 0.25$
F = 0.8
F = 0.6
1
0
−1
θ_c
$\beta = 30°$
figure 15

Figure 1 : Definition of the frames

Figure 2 : Existence of intersections of curves representing the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos\beta} - 1)$ and $-\sqrt{1+4\nu\cos\theta}$ according to the values of ν and β.

Figure 3 : Existence of intersections of curves representing the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos\beta} - 1)$ and $+\sqrt{1+4\nu\cos\theta}$ according to the values of ν and β.

Figure 4 : Existence of intersections of curves representing the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos} - 1)$ and $-\sqrt{1-4\nu\cos\theta}$ according to the values of $\nu < \nu_c$ and β.

Figure 5 : Existence of intersections of curves representative of the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos\beta} - 1)$ and $-\sqrt{1-4\nu\cos\theta}$ according to the values $\nu < \nu_c$ and β.

Figure 6 : Existence of intersections of curves representative of the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos} - 1)$ and $+\sqrt{1-4\nu\cos\theta}$ according to the values of $\nu < \nu_c$ and β.

Figure 7 : Existence of intersections of curves representative of the functions $1/2\ (\frac{\cos(2\theta-\beta)}{\cos\beta} - 1)$ and $+\sqrt{1-4\nu\cos\theta}$ according to the values of $\nu < \nu_c$ and β.

Figure 8 : All results relative to the existence of zeros of the functions $g'_i(\theta)$ $(i \in [1,4])$ in relation to ν and β.